

Microscopic Spectrum of the QCD Dirac Operator in Three Dimensions

Richard J. Szabo

*Department of Mathematics
Heriot-Watt University
Riccarton, Edinburgh EH14 4AS, Scotland
richard@ma.hw.ac.uk*

Abstract

The microscopic spectral correlators of the Dirac operator in three-dimensional Yang-Mills theory coupled to fundamental fermions and with three or more colours are derived from the supersymmetric formulation of partially quenched effective Lagrangians. The flavour supersymmetry breaking patterns are appropriately identified and used to calculate the corresponding finite volume partition functions from Itzykson-Zuber type integrals over supersymmetric cosets. New and simple determinant expressions for the spectral correlators in the mesoscopic scaling region are thereby found. The microscopic spectrum derived from the effective finite volume partition function of three-dimensional QCD agrees with earlier results based on the unitary ensemble of random matrix theory and extends the corresponding calculations for QCD in four dimensions.

1 Introduction and Summary

Quantum chromodynamics in three spacetime dimensions (QCD₃) provides an interesting and sometimes solvable testing ground for phenomena which occur in its four dimensional counterpart. It is related to the high temperature limit of QCD₄ [1] and also to quantum antiferromagnetism [2]. In this paper we will study some aspects of the spontaneous breaking of flavour symmetry in three dimensional QCD [1],[3]–[5]. This mechanism is the analog of the breakdown of chiral symmetry in four dimensions which is believed to be an important property of strong interactions.

QCD₃ with massless quarks in the fundamental representation of the $SU(N_c)$ gauge group, with $N_c \geq 3$, possesses a continuous, global flavour symmetry group $U(N_f)$ which acts on N_f species of two-component, parity-odd complex fermion fields ψ_i as $\psi_i \mapsto U_i^j \psi_j$, $\bar{\psi}^i \mapsto U_j^\dagger{}^i \bar{\psi}^j$, where $U \in U(N_f)$. Symmetry breaking occurs when there is an even number $N_f = 2n_f$ of fermion flavours [3]. To understand the mechanism, one introduces a small fermion mass term $i\bar{\psi}\mathcal{M}\psi$ with mass matrix \mathcal{M} which preserves the parity symmetry of the massless Euclidean field theory but which *explicitly* breaks the flavour symmetry. This can be achieved, for example, by arranging the quark masses m_i into pairs of opposite sign in the diagonal mass matrix¹

$$\mathcal{M} = \text{diag} \left(m_1, \dots, m_{n_f}, -m_1, \dots, -m_{n_f} \right) . \quad (1.1)$$

In this case, the corresponding fermion determinant is positive definite and one may invoke the Vafa-Witten theorem [6] to argue that, if flavour symmetry is spontaneously broken, then the diagonal elements of the $N_f \times N_f$ Hermitian fermion condensate matrix

$$\Sigma_j^i = \langle 0 | \bar{\psi}^i \psi_j | 0 \rangle \quad (1.2)$$

are equal in magnitude and of the same signs as the corresponding masses. This implies that the global flavour symmetry group is broken according to

$$U(2n_f) \longrightarrow U(n_f) \times U(n_f) \quad (1.3)$$

by the order parameter

$$\Sigma_0 = \frac{1}{2n_f} \text{tr} |\Sigma| . \quad (1.4)$$

In this instance, the discrete symmetry group \mathbb{Z}_2 , generated by the product of three-dimensional spacetime parity and the flavour exchange $\psi_i \leftrightarrow \psi_{n_f+i}$, $i = 1, \dots, n_f$, is unbroken and remains a good symmetry even at the quantum level.

¹Recall that in three Euclidean spacetime dimensions the (Dirac) fermion mass term is purely imaginary and odd under parity, $\bar{\psi}\psi \mapsto -\bar{\psi}\psi$. Fermion masses may therefore be positive or negative in three dimensions and change sign under parity.

On the other hand, by an analog of the Banks-Casher relation [7], the distribution of small eigenvalues of the three-dimensional Euclidean Dirac operator $i\mathcal{D}$ is related to the condensate (1.4) by

$$\Sigma_0 = \frac{\pi \rho(0)}{V} , \quad (1.5)$$

where $\rho(\lambda; m_1, \dots, m_{N_f})$ is the spectral density of $i\mathcal{D}$ and V is the volume of three-dimensional spacetime. Understanding the function ρ is therefore tantamount to a detailed description of the dynamics underlying flavour symmetry breaking in QCD₃. This distribution is difficult to compute in general, even near the spectral origin $\lambda = 0$. However, since only the low momentum modes of the Dirac operator spectrum are relevant, one may propose that ρ could be computed in the ergodic regime where the zero momentum mode of the corresponding Goldstone field U dominates [8]. In this case, the effective, finite-volume partition function becomes remarkably simple, and it is equivalent to the representation of QCD₃ in terms of microscopic degrees of freedom precisely in the limit of zero momentum. The spacetime integration over the effective Lagrangian produces an overall volume factor V , and the partition function simplifies to a finite dimensional group integral of the zero modes of U over the coset space determined by the pattern of flavour symmetry breaking. For the symmetry breakdown (1.3), we have

$$Z_{N_f}^{\text{LS}}(\mathcal{M}) = \int_{\mathcal{G}(n_f)} DU \, e^{-iV\Sigma_0 \text{tr}(\mathcal{M}U\Gamma_5 U^\dagger)} , \quad (1.6)$$

where $\Gamma_5 = \mathbb{1}_{n_f} \otimes \sigma_3$ ($\mathbb{1}_{n_f}$ denotes the $n_f \times n_f$ identity matrix and σ_3 the 2×2 diagonal Pauli spin matrix), DU denotes the invariant Haar measure on the $N_f \times N_f$ unitary group $U(N_f)$, and

$$\mathcal{G}(n_f) = \frac{U(2n_f)}{U(n_f) \times U(n_f)} \quad (1.7)$$

is the corresponding Goldstone manifold. The beauty of the expression (1.6) comes from the observation [9] that the integration may be extended from the symmetric space (1.7) to the group manifold of $U(2n_f)$. This follows from the fact that the subgroup of the flavour symmetry group whose adjoint action leaves the matrix Γ_5 invariant is precisely $U(n_f) \times U(n_f)$, so that the two integrals agree up to the volume of this stability subgroup in the Haar measure DU . The finite volume partition function (1.6) may then be evaluated analytically using the Itzykson-Zuber formula [10], and thereby used to explicitly derive quantities such as spectral sum rules for QCD₃.

One is ultimately interested in taking the thermodynamic limit $V \rightarrow \infty$ and the quark masses $m_i \rightarrow 0$. In the ergodic regime, one must keep the linear dimension $V^{1/3}$ of the system much smaller than the Compton wavelength of the Goldstone bosons, which is tantamount to holding fixed the parameters [8]

$$\omega_i = V \Sigma_0 m_i . \quad (1.8)$$

This approximation ensures that the non-zero momentum modes factorize from the effective Euclidean QCD₃ partition function. The crucial observation made some time ago [9] (see [11] for a recent review) was that the effective partition function (1.6) can be equivalently described by the large N limit of an $N \times N$ unitary random matrix ensemble. The simplest matrix model of this type is defined by the partition function²

$$Z_{N_f}^{\text{GUE}}(m_1, \dots, m_{N_f}) = \int_{u(N)} DT \, e^{-\frac{N\Sigma_0^2}{2} \text{tr} T^2} \prod_{j=1}^{N_f} \det(T - im_j) \quad (1.9)$$

of the Gaussian unitary ensemble, where DT is the Gaussian-normalized Haar measure on the Lie algebra $u(N)$ of $N \times N$ Hermitian matrices. Since the massless Euclidean Dirac operator $i\not{D}$ in an arbitrary background field is Hermitian, the matrix model (1.9) possesses the same global symmetries as the original field theory. The spacetime volume V translates directly into the size N of the matrices. The main conjecture put forward in this context is that the Dirac operator spectrum can be computed from the spectral correlation functions of the matrix model (1.9). With the *assumption* that the spectral properties of random matrix theory carry over to QCD₃, the quantity $\pi/\Sigma_0 V = 1/\rho(0)$ is the mean level spacing between the smallest eigenvalues of the Dirac operator, where $\rho(\lambda; m_1, \dots, m_{N_f})$ may now be computed as the spectral density of the Hermitian matrix T in the ensemble (1.9). The ergodic limit of QCD₃ thus becomes the microscopic or local scaling limit of the random matrix model (1.9), i.e. $N \rightarrow \infty$ with Nm_i fixed. This special limit motivates the introduction of a microscopic spectral density defined in the mesoscopic scaling region by

$$\rho_s(u; \omega_1, \dots, \omega_{N_f}) = \lim_{N \rightarrow \infty} \frac{1}{N\Sigma_0} \rho\left(\frac{u}{N\Sigma_0}; \frac{\omega_1}{N\Sigma_0}, \dots, \frac{\omega_{N_f}}{N\Sigma_0}\right), \quad (1.10)$$

where $u = \pi\rho(0)\lambda$ are the unfolded Dirac operator eigenvalues. For broken flavour symmetry the quantity (1.10) is a non-trivial function, and the order parameter (1.4) may be computed from the resolvent function of the ensemble (1.9) as

$$\Sigma_0 = -i \lim_{\mathcal{M} \rightarrow 0} \lim_{N \rightarrow \infty} \frac{1}{N} \frac{\partial}{\partial m_i} \ln Z_{N_f}^{\text{GUE}}(m_1, \dots, m_{N_f}), \quad (1.11)$$

for any $i = 1, \dots, n_f$.

The microscopic spectral density (1.10) has been computed using random matrix theory techniques in [9, 10] and related aspects of QCD₃ within this framework are described in [13]–[15]. However, in order to have a better understanding of the relationship between the effective field theory (1.6) and the random matrix theory (1.9) in the microscopic domain, one would like to compute the spectral density directly from the low-energy effective field theory. Indeed, the calculation of the Dirac operator spectrum should not

²The universality of random matrix theory results, i.e. the insensitivity to the details of the particular matrix potential in the appropriate limit, is well established [12]. In this paper we will consider the simplest Gaussian potentials, consistent with only the general symmetries of the problem as input.

rely solely on random matrix theory techniques, and should follow directly from quantum field theory. This is a non-trivial computation because the usual infrared limit of the QCD₃ partition function, which is dominated by the Goldstone modes associated with the spontaneous flavour symmetry breaking, does not access the Dirac operator spectrum. If possible though, the matching of such results with those of random matrix theory would constitute a direct *proof* that the universal matrix model calculations do indeed reproduce the microscopic Dirac operator spectrum of QCD₃.

The problem was solved for four-dimensional QCD in [17] by the introduction of a species of fictitious “valence” quarks, paired with yet another set of fictitious particles of opposite quantum mechanical statistics. This leads to a model of “partially quenched” QCD₃ [18] containing N_v valence quarks and their supersymmetric partners, of masses μ_i and $\bar{\mu}_i$, $i = 1, \dots, N_v$, respectively, and N_f unquenched (physical) sea quarks of masses m_i , $i = 1, \dots, N_f$. The quark fields are all assumed to transform in the fundamental representation of the gauge group. In the original field theory formulation, the Euclidean partition function is

$$Z_{N_f, N_v}(\{m_i\}; \{\mu_i, \bar{\mu}_i\}) = \int [dA] \prod_{i=1}^{N_v} \frac{\det(i\mathcal{D} - i\mu_i)}{\det(i\mathcal{D} - i\bar{\mu}_i)} \prod_{j=1}^{N_f} \det(i\mathcal{D} - im_j) e^{-S_{\text{YM}}[A]} \quad (1.12)$$

where $S_{\text{YM}}[A]$ is the three-dimensional Yang-Mills action. When $\mu_i = \bar{\mu}_i$ for each $i = 1, \dots, N_v$, the fermion determinants arising from integration over the valence quarks are cancelled by the contributions from the corresponding bosonic ghost quarks of the same masses. Then, the partition function (1.12) reduces to that of ordinary QCD₃ with N_f physical flavours of fermions. We shall refer to this case as the “supersymmetric limit”, but we will keep the masses generically distinct to lift the degeneracy between the valence quarks and their superpartners. The partition function (1.12) is now also the generating function for mass-dependent condensates of the extra quark species as [19]

$$\begin{aligned} \Sigma_s(i\mu_j; \omega_1, \dots, \omega_{N_f}) &= -\frac{i}{N} \frac{\partial}{\partial \mu_j} \ln Z_{N_f, N_v} \left(\left\{ \frac{\omega_i}{N\Sigma_0} \right\}; \{\mu_i, \bar{\mu}_i\} \right) \Big|_{\{\mu_i = \bar{\mu}_i\}} \\ &= -2i \Sigma_0 \mu_j \int_0^\infty du \frac{\rho_s(u; \omega_1, \dots, \omega_{N_f})}{u^2 - N^2 \Sigma_0^2 \mu_j^2}, \end{aligned} \quad (1.13)$$

where the second equality follows from the spectral representation of the condensate and the fact that ρ_s is an even function of u . In this equation ρ_s is the microscopic spectral density of the Dirac operator in the original, unquenched field theory. The condensate in (1.13) can be expressed as the Stieltjes transform of $\rho_s(u; \omega_1, \dots, \omega_{N_f})$ which, under suitable convergence criteria for the function Σ_s , has a unique inverse given by the discontinuity of (1.13) across the real axis,

$$\rho_s(u; \omega_1, \dots, \omega_{N_f}) = \frac{1}{2\pi i \Sigma_0} \lim_{\epsilon \rightarrow 0^+} \left[\Sigma_s(u + i\epsilon; \omega_1, \dots, \omega_{N_f}) - \Sigma_s(u - i\epsilon; \omega_1, \dots, \omega_{N_f}) \right]. \quad (1.14)$$

Therefore, a detailed understanding of the partially quenched partition functions (1.12) will enable a precise, field theoretical determination of the microscopic spectral density of the Dirac operator for QCD₃.

In this paper we will discuss how to evaluate the microscopic spectrum of the QCD₃ Dirac operator using the partially quenched quantum field theory (1.12), thereby extending the computations in four dimensions [17]. In doing so, we will uncover some subtleties concerning the breaking of flavour symmetry in these models. The main observation we shall make may be summarized as follows. The basic flavour symmetry of the QCD₃ action with N_f sea quarks and N_v valence quarks is parametrized by the Lie supergroup $GL(N_f + N_v | N_v)$. As we are ultimately interested in studying the microscopic density of Dirac operator eigenvalues corresponding to a broken symmetry phase, we assume that $N_f = 2n_f$ and work with the parity-symmetric mass matrix (1.1). Let us further assume that there are n_v^+ positive masses $\bar{\mu}_i$ and n_v^- negative ones, so that $N_v = n_v^+ + n_v^-$. We will see that in this case the flavour supersymmetry in the massless limit is broken according to

$$GL(2n_f + n_v^+ + n_v^- | n_v^+ + n_v^-) \longrightarrow GL(n_f + n_v^+ | n_v^+) \times GL(n_f + n_v^- | n_v^-) . \quad (1.15)$$

The details of the symmetry breaking pattern (1.15) present some subtleties in the computation of the spectral density via the expression (1.13,1.14). The low momentum, finite volume partition function corresponding to the field theory (1.12) is a supersymmetric generalization of the Itzykson-Zuber integral (1.6) taken over the Goldstone supermanifold

$$\hat{\mathcal{G}}(n_f; n_v^+, n_v^-) = \frac{GL(2n_f + n_v^+ + n_v^- | n_v^+ + n_v^-)}{GL(n_f + n_v^+ | n_v^+) \times GL(n_f + n_v^- | n_v^-)} . \quad (1.16)$$

However, in contrast to the unquenched case, one cannot extend the coset (1.16) to the full flavour supergroup in a straightforward way, because the volume of the unitary supergroup vanishes in its Haar-Berezin measure [21]. Therefore, the finite volume partition function in this case must be dealt with as an integral over a coset superspace, rather than a Lie supergroup. This makes the parametrization of the integration variables far more intricate, and there is no known generalization of the Itzykson-Zuber formula for such superspaces (The generalization of the Itzykson-Zuber formula for the unitary supergroup has been derived in [22]). These subtleties do not arise in the four dimensional case, as the patterns of chiral symmetry breaking in both quenched and unquenched cases are completely analogous, and the effective finite volume field theory is given by an integral over a Lie supergroup which is the straightforward supersymmetric generalization of that for the unquenched case [17].

For example, consider the case of only a single species of valence quarks, $N_v = 1$, which is the pertinent partially quenched model from which to extract the spectral density. Then, according to (1.15), the flavour symmetry breaking pattern is

$$GL(2n_f + 1 | 1) \longrightarrow U(n_f) \times GL(n_f + 1 | 1) . \quad (1.17)$$

It follows that there is no symmetry breaking associated with the flavour supersymmetry of the theory, only that which is associated with the original unquenched field theory. While the symmetry breaking pattern (1.17) complicates the evaluation of the function ρ_s from (1.13,1.14), it will turn out to be the correct answer which gives the spectral distributions in the microscopic scaling limit that are anticipated from random matrix theory. This will provide an analytical demonstration that the microscopic distribution of eigenvalues of the QCD Dirac operator in three dimensions can be computed from an intricate, supersymmetric extension of the effective finite-volume QCD₃ partition function (1.6), i.e. that the Dirac operator spectrum in three dimensions can be derived directly from quantum field theory. In addition, it yields an analytic proof that the smallest eigenvalues of the QCD₃ Dirac operator are correlated according to a random matrix model whose form is dictated by the global symmetries of $i\not{D}$. In this way we will present the appropriate generalization of the results for four spacetime dimensions and the chiral unitary ensemble of random matrix theory to three spacetime dimensions and the ordinary unitary ensemble.

The organization of the remainder of this paper is as follows. In section 2 we present some field theoretical arguments for the symmetry breaking patterns (1.15). In section 3 these same patterns are derived using a random matrix theory representation of the quantum field theory (1.12), along with the finite volume, low energy effective field theory in the local scaling limit. In section 4 we illustrate these formal properties of the partition functions (1.12) by performing some explicit calculations in the quenched approximation. In section 5 we present the calculation of the microscopic spectrum of the Dirac operator. There we derive new expressions for the spectral density ρ_s which agree with those previously derived in the literature using random matrix theory, but which are much more compact and useful. As an interesting by-product of this analysis, we will also uncover an elegant representation of the finite volume partition function (1.6) itself. In section 6 we extend this analysis to compute all microscopic k -point correlation functions. Two appendices at the end of the paper contain some technical aspects of our analysis. In appendix A we formally prove that the low-energy effective field theory reduces exactly to (1.6) in the supersymmetric limit, as it should. In appendix B we present a simple and self-contained derivation of the Itzykson-Zuber formula for the unitary supergroup [22] which is used in the calculations of sections 5 and 6.

2 Flavour Symmetry Breaking in Three Dimensions

In this section we will begin our analysis of the microscopic regime of QCD₃ by presenting some heuristic, field theoretical arguments for the patterns of flavour symmetry breaking in the partially quenched models (1.12). In this paper we will deal only with the case of an even number N_f of physical fermion flavours. For an odd number of physical flavours, the discrete \mathbb{Z}_2 symmetry of the theory, composed of parity and flavour exchange, is

broken explicitly in the massive case, while for massless quarks it is broken radiatively by a gauge invariant anomaly which manifests itself in the appearance of a Chern-Simons term at one-loop order [4]. For even N_f this anomaly vanishes, and so we shall henceforth work with this case to facilitate some of the arguments which follow.

Consider partially quenched, Euclidean QCD₃ with $N_c \geq 3$ colours, N_f flavours of fundamental sea quarks, and N_v flavours of fundamental valence quarks. As mentioned in the previous section, the tree-level global symmetry group of this quantum field theory is $GL(N_f + N_v | N_v)$. This group rotates the fermionic sea and valence quarks, and also the bosonic superpartners of the valence quarks, among each other. To define the quantum path integral, we need to choose a measurable subspace of it. The maximally symmetric Riemannian submanifold of $GL(N_f + N_v | N_v)$ is supported by the ordinary, compact Lie group $U(N_f + N_v)$ in the fermion-fermion sector, and by the non-compact Lie group $GL(N_v, \mathbb{C})/U(N_v)$ in the boson-boson sector. While the Grassmann integrations are well-defined in the path integral representations (1.12), convergence of the integrations over the bosonic quark fields is inconsistent with compact $U(N_v)$ flavour rotations in this sector. For this reason, the bosonic valence quarks must transform under a non-compact group of flavour transformations which is consistent with convergence requirements. We shall meet this requirement again in a somewhat more explicit form in the next section. This structure is necessary to produce a positive definite quadratic form for the kinetic and mass terms of the low-energy effective Lagrangian.

We will begin by deducing the symmetry breaking pattern (1.15) for the massless quantum field theory by employing a generalization of the Coleman-Witten argument for ordinary QCD₄ [23]. For this, it is instructive to first deduce the pattern (1.3) in the original, unquenched massless field theory, a possibility which was mentioned in [9]. Under a global rotation in flavour space, the Hermitian fermion condensate matrix (1.2), which is a natural order parameter for flavour symmetry breaking, transforms as

$$\Sigma \longmapsto U^\dagger \Sigma U \quad , \quad U \in U(N_f) \quad , \quad (2.1)$$

while under a \mathbb{Z}_2 parity transformation it maps as

$$\Sigma \longmapsto -\Sigma \quad . \quad (2.2)$$

We assume that all of the criteria of [23] apply in our case. In particular, the effective potential V_f , obtained by integrating out the Yang-Mills fields, is a $U(N_f) \times \mathbb{Z}_2$ invariant function of the condensate matrix Σ which does not display any accidental degeneracy with respect to the flavour and parity symmetries of the theory. This means that any ground state of the quantum field theory, obtained by minimizing the effective potential $V_f(\Sigma)$, can be obtained from any other one by a $U(N_f) \times \mathbb{Z}_2$ transformation.

From the continuous symmetry (2.1) it follows that V_f is a function only of the N_f real-valued eigenvalues $\sigma_1, \dots, \sigma_{N_f}$ of the condensate matrix Σ . From the discrete symmetry (2.2), it follows that the effective potential is an even function of the eigenvalues,

$$V_f(-\sigma_1, \dots, -\sigma_{N_f}) = V_f(\sigma_1, \dots, \sigma_{N_f}) \quad . \quad (2.3)$$

The simplification we would now like to make is to take the limit $N_c \rightarrow \infty$ of a large number of colours. It is well-known that in this limit only the contributions from planar 't Hooft diagrams survive [24]. These graphs correspond to the Feynman diagrams that contain only a single quark loop which, in the expansion of the invariant function V_f in powers of traces of powers of the condensate matrix, are generated by single trace insertions of powers of Σ , i.e.

$$V_f(\Sigma) = \text{tr } \mathcal{V}_f(\Sigma) + \mathcal{O}\left(\frac{1}{N_c}\right) = \sum_{i=1}^{N_f} \mathcal{V}_f(\sigma_i) + \mathcal{O}\left(\frac{1}{N_c}\right), \quad (2.4)$$

where \mathcal{V}_f is some scalar function which is independent of N_c . Since the eigenvalues σ_i are independent variables, the ground states are determined by minimizing each term $\mathcal{V}_f(\sigma_i)$ in (2.4). Thus each eigenvalue of Σ is at the minimum of the function \mathcal{V}_f . From the reflection symmetry (2.3) it follows that if σ_i is a minimum, then so is $-\sigma_i$. We conclude that either all eigenvalues vanish, or else there are $n_f = N_f/2$ strictly positive and equal eigenvalues σ_i with the n_f other ones being their \mathbb{Z}_2 reflections. The first possibility, which corresponds to the case of unbroken flavour symmetry, may be excluded by arguing similarly to [23]. Namely, the three-current Green's function contains only massless poles in the large N_c limit and so the fermion bilinear current must create a massless scalar particle from the vacuum state. By Goldstone's theorem, this implies the spontaneous breakdown of the continuous symmetry, and so only the second possibility remains. In this way we deduce the symmetry breaking pattern (1.3).

Now let us consider the partially quenched quantum field theory (1.12) with massless fields. We define a condensate matrix $\hat{\Sigma}$ similarly to (1.2). It is a supermatrix which lives in the Lie superalgebra of the flavour supergroup $GL(N_f + N_v | N_v)$ and which transforms under the adjoint action of global flavour superrotations. Since the real-valued eigenvalues of $\hat{\Sigma}$ are commuting variables, we can go through the above argument by considering individually the fermion-fermion and boson-boson blocks of the theory. The total, effective potential for the partially quenched model may then be written as

$$V(\hat{\Sigma}) = \sum_{i=1}^{N_f} \mathcal{V}_f(\sigma_i) + \sum_{i=1}^{N_v} \mathcal{V}_v(\xi_i) + \sum_{\alpha=1}^{N_v} \bar{\mathcal{V}}_v(\bar{\xi}_\alpha) + \mathcal{O}\left(\frac{1}{N_c}\right), \quad (2.5)$$

where \mathcal{V}_f is, as above, the contribution from Feynman diagrams consisting of only a single physical quark loop, \mathcal{V}_v is the contribution from single valence quark loops, and $\bar{\mathcal{V}}_v$ comes from single loops of the bosonic ghost quark fields. The ground state is now determined by independently minimizing each of the terms $\mathcal{V}_f(\sigma_i)$, $\mathcal{V}_v(\xi_i)$ and $\bar{\mathcal{V}}_v(\bar{\xi}_\alpha)$ in (2.5). The first term is minimized as explained above.

For the last two terms, we recall that for our purposes the valence quarks are fictitious particles and we are really interested in the near massless limit of the quantum field theory (1.12). Let us assume that there are n_v^+ positive masses $\bar{\mu}_i$ and n_v^- negative ones. Since we are ultimately interested in taking the supersymmetric limit $\mu_i = \bar{\mu}_i$, the same structure is assumed to hold for the fermionic valence quark masses μ_i . Again

convergence requirements restrict the flavour symmetry of the ghost quark fields to a non-compact $U(n_v^+, n_v^-)$ subgroup of $GL(N_v, \mathbb{C})$, which defines a maximally symmetric Riemannian submanifold. Let $n_v = \min(n_v^+, n_v^-)$. Since the bosonic ghost quark fields transform as scalars under spacetime parity, it follows that the action in (1.12) has a “reduced” \mathbb{Z}_2 parity symmetry defined by reflecting n_v of the masses of a given sign into n_v masses of the opposite sign and interchanging the n_v pairs of corresponding flavours. In terms of the eigenvalues of $\hat{\Sigma}$ this is represented as the symmetry

$$\bar{\xi}_\alpha \longleftrightarrow -\bar{\xi}_{\alpha+n_v} \quad , \quad 1 \leq \alpha \leq n_v \quad (2.6)$$

of the effective potential (2.5) (We have used the fact that the eigenvalues in the large N_c limit may be arbitrarily ordered). Applying the above reasoning to the minima of $\bar{\mathcal{V}}_v$ we conclude that either all eigenvalues $\bar{\xi}_\alpha$ vanish in the ground state, or else there are $N_v - 2n_v$ non-vanishing equal eigenvalues, n_v of which map onto the remaining n_v eigenvalues by a \mathbb{Z}_2 parity transformation. Excluding the first possibility, this breaks the $U(n_v^+, n_v^-)$ flavour symmetry to the subgroup $U(n_v^+) \times U(n_v^-)$. By supersymmetry, we may also infer the symmetry breaking $U(N_v) \rightarrow U(N_v - n_v) \times U(n_v)$ for the valence quarks. Since the pattern of symmetry breaking is determined entirely by the pattern of eigenvalues at the minimum of the effective potential, we arrive at the flavour supersymmetry breaking pattern (1.15).

Generally, by a supersymmetric generalization of the Vafa-Witten theorem [6] and the appropriate assumption of spontaneous flavour supersymmetry breaking, one may arrive at the pattern (1.15). The low energy physics of QCD₃ is completely determined by the spontaneous symmetry breaking pattern described above [8]. The effective field theory for the low momentum modes of the Goldstone superfield may now be derived analogously to the four dimensional case which utilizes chiral perturbation theory [17, 18]. In the ergodic regime, the effective Lagrangian in Euclidean space is given by

$$\mathcal{L}_{\text{eff}} = \frac{f_\pi^2}{4} \text{STr} \partial_\mu \mathcal{U} \partial_\mu \mathcal{U}^{-1} - \frac{i \Sigma_0}{2} \text{STr} \left| \hat{\mathcal{M}}' \right| \left(\mathcal{U} + \mathcal{U}^{-1} \right) , \quad (2.7)$$

where f_π is the pion decay constant, $\hat{\mathcal{M}}'$ is the mass matrix of the field theory (1.12), and the supertrace STr will be defined in the next section. The masses of the Goldstone bosons are given by the usual Gell Mann-Oakes-Renner relation $M_{AB}^2 = (\hat{\mathcal{M}}_{AA} + \hat{\mathcal{M}}_{BB}) \Sigma_0 / f_\pi^2$. The corresponding superfields \mathcal{U} live in the vacuum manifold for the symmetry breakdown and may be parametrized as

$$\mathcal{U} = U \text{diag} \left(\mathbb{1}_{n_f+n_v^+}, -\mathbb{1}_{n_f+n_v^-} \middle| \mathbb{1}_{n_v^+}, -\mathbb{1}_{n_v^-} \right) U^{-1} \quad (2.8)$$

with $U \in \hat{\mathcal{G}}(n_f; n_v^+, n_v^-)$. This leads to a supersymmetric generalization of the finite volume partition function (1.6) with integration domain the symmetric superspace (1.16).

A tantalizing aspect of the symmetry breaking here is the possibility of unbroken flavour supersymmetry in the valence sector, which occurs whenever $n_v = 0$, i.e. all valence

quark masses are either positive or negative, thereby destroying the parity symmetry of the valence fields. Such a situation arises, for example, in the fully quenched case $N_f = 0$ with a single species of valence quarks. Then, the $GL(1|1)$ flavour supersymmetry is unbroken. In that case, a very simple argument [8] is sufficient to determine the precise form of the partition function (1.12) in the microscopic limit. For this, we expand the vacuum energy in powers of the masses μ and $\bar{\mu}$ by treating the mass terms as perturbations. Since there is no spontaneous breaking of the continuous flavour symmetry, there are no massless particles in the spectrum of the quantum field theory, and hence the perturbation series does not produce any infrared divergences. The vacuum energy per unit spacetime volume thereby admits the Taylor series expansion

$$-\frac{1}{N} \ln Z_{0,1}(\mu, \bar{\mu}) = z_0 + z_1 \mu + \bar{z}_1 \bar{\mu} + \sum_{n+m \geq 2} z_{nm} \mu^n \bar{\mu}^m . \quad (2.9)$$

The constant z_0 affects only the overall normalization of the fully quenched partition function and may be set to zero. In the microscopic limit, the masses are taken to vary as $\mu, \bar{\mu} \sim \frac{1}{N}$. In the thermodynamic limit, the infinite sum in (2.9) therefore vanishes and we have

$$-\ln Z_{0,1}(\mu, \bar{\mu}) = N z_1 \mu + N \bar{z}_1 \bar{\mu} + \mathcal{O}\left(\frac{1}{N}\right) . \quad (2.10)$$

The constants z_1 and \bar{z}_1 may be determined from the definitions (1.4) and (1.13) to be

$$\begin{aligned} z_1 &= -\frac{1}{N} \left. \frac{\partial \ln Z_{0,1}(\mu, \bar{\mu})}{\partial \mu} \right|_{\mu=\bar{\mu}=0} = +i \Sigma_0 \operatorname{sgn} \mu , \\ \bar{z}_1 &= -\frac{1}{N} \left. \frac{\partial \ln Z_{0,1}(\mu, \bar{\mu})}{\partial \bar{\mu}} \right|_{\mu=\bar{\mu}=0} = -i \Sigma_0 \operatorname{sgn} \mu , \end{aligned} \quad (2.11)$$

where we have used the standard convention (dictated by the Vafa-Witten theorem) that the fermion condensate $\langle 0 | \bar{\psi} \psi | 0 \rangle$ and the corresponding quark mass are of the same sign. In this way the mass dependence of the fully quenched QCD₃ partition function in the microscopic limit may be explicitly computed to be

$$Z_{0,1}^{(\infty)}(\mu, \bar{\mu}) = e^{-i N \Sigma_0 (\mu - \bar{\mu}) \operatorname{sgn} \mu} . \quad (2.12)$$

Later on we shall see that the simple form (2.12) agrees with what one obtains from the generic finite-volume field theory and also from random matrix theory arguments.

3 Low Energy Effective Field Theory

In the previous section we presented arguments in favour of a rather intricate pattern of flavour supersymmetry breaking in partially quenched QCD₃. This symmetry breaking pattern is crucial for the effective field theory computation of the microscopic spectral density. In the four dimensional supersymmetric formulation [17], the symmetry breaking

pattern is *assumed* to mimic as closely as possible the known bosonic one, and this turns out to be the correct answer. To help clarify the origin of the required finite volume partition functions, in this section we will derive the low-energy effective field theory for partially quenched QCD₃ in the microscopic domain using the supersymmetry technique of random matrix theory [20]. Although the main subject of this paper is to examine properties of the Dirac operator spectrum using field theory methods, it will prove instructive to use random matrix theory as an intermediate step. This will provide, following [25], a rigorous derivation of the symmetry breaking arguments of the previous section. For instance, it will clarify how to choose the appropriate integration domain, for the definition of the partition function, as a Riemannian submanifold of the Goldstone supermanifold (1.16).

For this, we start by writing down a random matrix model with the same global symmetries as the field theory (1.12) in the Gaussian unitary ensemble. The partition function is

$$Z_{N_f, N_v}(\{m_i\}; \{\mu_i, \bar{\mu}_i\}) = \int_{u(N)} DT \, e^{-\frac{N\Sigma_0^2}{2} \text{tr} T^2} \prod_{i=1}^{N_v} \frac{\det(T - i\mu_i)}{\det(T - i\bar{\mu}_i)} \prod_{j=1}^{N_f} \det(T - im_j) . \quad (3.1)$$

However, the matrix model (3.1) is difficult to deal with directly because of the asymmetry between the valence and sea sectors. It will prove convenient throughout this paper to have a completely supersymmetric form of the field theory, even in the physical sector. To this end we introduce a set of bosonic sea quarks which are the superpartners of the dynamical fermions. They have masses \bar{m}_i , $i = 1, \dots, N_f$, which by supersymmetry are distributed according to sign as in (1.1) in the mass matrix

$$\bar{\mathcal{M}} = \text{diag}(\bar{m}_1, \dots, \bar{m}_{n_f}, -\bar{m}_1, \dots, -\bar{m}_{n_f}) . \quad (3.2)$$

The matrix model partition function of this extended supersymmetric field theory is defined by³

$$\mathcal{Z}_{N_f, N_v}(\{m_i, \bar{m}_i\}; \{\mu_i, \bar{\mu}_i\}) = \int_{u(N)} DT \, e^{-\frac{N\Sigma_0^2}{2} \text{tr} T^2} \prod_{i=1}^{N_v} \frac{\det(T - i\mu_i)}{\det(T - i\bar{\mu}_i)} \prod_{j=1}^{N_f} \frac{\det(T - im_j)}{\det(T - i\bar{m}_j)} , \quad (3.3)$$

and the desired partition function (3.1) may then be computed from the infrared regime of the bosonic sea sector of (3.3),

$$Z_{N_f, N_v}(\{m_i\}; \{\mu_i, \bar{\mu}_i\}) = \prod_{j=1}^{N_f} \lim_{\bar{m}_j \rightarrow \infty} (-i\bar{m}_j)^N \mathcal{Z}_{N_f, N_v}(\{m_i, \bar{m}_i\}; \{\mu_i, \bar{\mu}_i\}) . \quad (3.4)$$

We will denote by $N_T = N_f + N_v$ the total number of flavours.

³In the following the notation \mathcal{Z} is strictly used for supersymmetric partition functions only.

3.1 Supersymmetric Representation

We will now apply a colour-flavour transformation [26] to rewrite the integration in (3.3) as an integral over the total flavour space of the valence and sea quarks. Let us introduce an $(N_T|N_T)$ complex supervector by

$$\Psi = \begin{pmatrix} \psi \\ \phi \end{pmatrix}, \quad (3.5)$$

where ψ_i , $i = 1, \dots, N_T$, are complex fermionic variables transforming in the vector representation of $U(N)$ which can be thought of as representing the valence and sea quarks, while ϕ_α , $\alpha = 1, \dots, N_T$, are complex bosonic variables, also in the vector representation of $U(N)$, which can be thought of as representing the ghost valence and sea quarks. The determinants appearing in (3.1) may then be exponentiated in the form

$$\begin{aligned} & \prod_{i=1}^{N_v} \frac{\det(T - i\mu_i)}{\det(T - i\bar{\mu}_i)} \prod_{j=1}^{N_f} \frac{\det(T - im_j)}{\det(T - i\bar{m}_j)} \\ &= (-1)^{n_f + n_v^-} \int_{(\mathbb{C}^{N_T|N_T})^N} D\Psi D\bar{\Psi} e^{\bar{\Psi} \wedge (iT \otimes \mathbb{1}_{(N_T|N_T)} + \mathbb{1}_N \otimes \hat{\mathcal{M}}) \Psi}, \end{aligned} \quad (3.6)$$

where

$$\hat{\mathcal{M}} = \text{diag} \left(\mathcal{M}, \mu_1, \dots, \mu_{N_v} \mid \bar{\mathcal{M}}, \bar{\mu}_1, \dots, \bar{\mu}_{N_v} \right) \quad (3.7)$$

is the mass matrix (ordered appropriately according to sign as in (1.1)), and the supervector measure is defined by

$$D\Psi D\Psi^\dagger = \prod_{a=1}^N \prod_{\alpha=1}^{N_T} \frac{d\phi_{\alpha,a} d\phi_{\alpha,a}^*}{\pi} \otimes \prod_{i=1}^{N_T} \frac{\partial}{\partial \psi_{i,a}} \frac{\partial}{\partial \psi_{i,a}^*}. \quad (3.8)$$

The symbol \wedge in (3.6) denotes the graded inner product of supervectors (3.5) defined by

$$\Psi^\dagger \wedge \Psi' = \psi^\dagger \psi' - \phi^\dagger \phi', \quad (3.9)$$

while the adjoint $\bar{\Psi}$ of the supervector (3.5) is defined as

$$\bar{\Psi} = \Psi^\dagger \Omega \quad \text{with} \quad \Omega = \text{diag} \left(\mathbb{1}_{N_T} \mid \mathbb{1}_{n_f}, -\mathbb{1}_{n_f}, \mathbb{1}_{n_v^+}, -\mathbb{1}_{n_v^-} \right). \quad (3.10)$$

The definition (3.10) ensures that the integration in (3.6) is uniformly convergent.

The integrations in both (3.3) and (3.6) are uniformly convergent. Interchanging them to perform the Gaussian integral over the Hermitian matrix T , we get

$$\begin{aligned} \mathcal{Z}_{N_f, N_v}(\{m_i, \bar{m}_i\}; \{\mu_i, \bar{\mu}_i\}) &= (-1)^{n_f + n_v^-} \int_{(\mathbb{C}^{N_T|N_T})^N} D\Psi D\bar{\Psi} \exp \left[\bar{\Psi} \wedge (\mathbb{1}_N \otimes \hat{\mathcal{M}}) \Psi \right. \\ &\quad \left. - \frac{1}{2N\Sigma_0^2} \text{STr} \begin{pmatrix} \sum_a \psi_a^\dagger \otimes \psi_a & -\sum_a \phi_a^\dagger \otimes \psi_a \\ \sum_a \psi_a^\dagger \otimes \phi_a & -\sum_a \phi_a^\dagger \otimes \phi_a \end{pmatrix}^2 \right]. \end{aligned} \quad (3.11)$$

Here STr denotes the supertrace on $(N_T|N_T)$ supermatrices defined by

$$\text{STr} \begin{pmatrix} A_{ff} & A_{bf} \\ A_{fb} & A_{bb} \end{pmatrix} = \text{tr } A_{ff} - \text{tr } A_{bb} , \quad (3.12)$$

where A_{ff} denotes the bosonic fermion-fermion block, A_{bb} the bosonic boson-boson block, and A_{bf}, A_{fb} the Grassmann boson-fermion blocks of the supermatrix. These blocks are all $N_T \times N_T$ matrices. The sums in (3.11) run from 1 to N .

To do the vector integration in (3.11), we rewrite the four vector interaction term in Gaussian form by using the generalized Hubbard-Stratonovich transformation

$$e^{-\frac{1}{2N\Sigma_0^2} \text{STr} (\sum_a \bar{\Psi}_a \otimes \Psi_a)^2} = \int_{gl(N_T|N_T)} D\Lambda \ e^{-\frac{N\Sigma_0^2}{2} \text{STr} \Lambda^2 + i \bar{\Psi} \Lambda (\mathbb{1}_N \otimes \Lambda) \Psi} , \quad (3.13)$$

where $D\Lambda$ is the invariant Haar-Berezin measure on the Lie superalgebra $gl(N_T|N_T)$. The elements of this superalgebra may be parametrized as

$$\Lambda = \begin{pmatrix} \lambda & \bar{\chi} \\ \chi & i\bar{\lambda} \end{pmatrix} \quad (3.14)$$

where $\lambda \in u(N_T)$ contains ordinary mesons made of quarks and antiquarks, χ and $\bar{\chi}$ are independent complex-valued Grassmann matrices representing fermionic mesons consisting of a ghost quark and an ordinary anti-quark, and the boson-boson block $i\bar{\lambda}$, which represents a meson constructed from two ghost quarks, parametrizes the Lie algebra of the non-compact group $GL(N_T, \mathbb{C})/U(N_T)$. Because of the supertraces, this compact/non-compact structure is required for convergence of the integration in (3.13). The normalized Berezin integration measure in (3.13) may be written in terms of the Cartesian coordinate parametrization (3.14) as

$$D\Lambda = (-\pi)^{-N_T^2/2} \prod_{i,j=1}^{N_T} d\lambda_{ij} \prod_{\alpha,\beta=1}^{N_T} d\bar{\lambda}_{\alpha\beta} \otimes \prod_{k=1}^{N_T} \prod_{\sigma=1}^{N_T} \frac{\partial}{\partial \chi_{k\sigma}} \frac{\partial}{\partial \bar{\chi}_{\sigma k}} . \quad (3.15)$$

However, the resulting integration over the supervector Ψ only converges when the boson-boson block $\bar{\lambda}$ of the supermatrix Λ takes values in a non-compact $u(n_f + n_v^+, n_f + n_v^-)$ subalgebra of $gl(N_T, \mathbb{C})$ defined by the condition $\Lambda^\dagger = \Omega \Lambda \Omega$.⁴ Then the integrations are all uniformly convergent, and so by integrating first over the supervector Ψ in (3.11,3.13) we arrive at

$$\mathcal{Z}_{N_f, N_v}(\{m_i, \bar{m}_i\}; \{\mu_i, \bar{\mu}_i\}) = \int_{gl(N_T|N_T)} D\Lambda \ e^{-\frac{N\Sigma_0^2}{2} \text{STr} \Lambda^2} \left[\text{SDet} (\Lambda - i \hat{\mathcal{M}}) \right]^N \quad (3.16)$$

where the superdeterminant is defined by

$$\text{SDet} \begin{pmatrix} A_{ff} & A_{bf} \\ A_{fb} & A_{bb} \end{pmatrix} = \frac{\det (A_{ff} - A_{bf} A_{bb}^{-1} A_{fb})}{\det A_{bb}} = \frac{\det A_{ff}}{\det (A_{bb} - A_{fb} A_{ff}^{-1} A_{bf})} \quad (3.17)$$

⁴See [25] for a more precise description of the entire integration domain.

and it satisfies $\text{STr } \ln \Lambda = \ln \text{SDet } \Lambda$. The expression (3.16), which is exact at the level of the random matrix model, is the desired representation of the partition function as an integral over the flavour superspace. Note that the partition function (3.16), in the supersymmetric limit and in the limit of massless sea quarks, is invariant under the transformations $\Psi \mapsto U \Psi$ which leave the bilinear form $\bar{\Psi} \wedge \Psi$ invariant. Such transformations satisfy the condition $U^\dagger = \Omega U^{-1} \Omega$ and form the unitary supergroup $U(N_{\text{T}}|n_f + n_v^+, n_f + n_v^-)$. This illustrates how the matrix model captures the global symmetries of the original field theory. In the next subsection we shall derive the symmetry breaking pattern from this point of view.

3.2 Local Scaling Limit

Let us now consider the partition function $\mathcal{Z}_{N_f, N_v}(\{m_i, \bar{m}_i\}; \{\mu_i, \bar{\mu}_i\})$ in the thermodynamic limit. Precisely, we want to study the superintegral (3.16) in the local scaling limit $N \rightarrow \infty$ whereby the quark masses are rescaled by the mean level spacing $\rho(0) = N\Sigma_0/\pi$ and held fixed. In this limit, the partition function (3.16) may be expanded according to

$$\begin{aligned} & \mathcal{Z}_{N_f, N_v}(\{m_i, \bar{m}_i\}; \{\mu_i, \bar{\mu}_i\}) \\ &= \int_{gl(N_{\text{T}}|N_{\text{T}})} D\Lambda \exp \text{STr} \left[-\frac{N\Sigma_0^2}{2} \Lambda^2 + N \ln \Lambda - \frac{i}{\Sigma_0} \Lambda^{-1} \hat{\mathcal{M}}_s + \mathcal{O}\left(\frac{1}{N}\right) \right] \end{aligned} \quad (3.18)$$

where $\hat{\mathcal{M}}_s = N\Sigma_0 \hat{\mathcal{M}}$, and we have implicitly restricted the integration in (3.18) to invertible Λ (This will be valid within a saddle-point approximation to follow). In the large N limit, the integral (3.18) is dominated by the stationary points of the function

$$\mathcal{F}(\Lambda) = \text{STr} \left(\frac{\Sigma_0^2}{2} \Lambda^2 - \ln \Lambda \right) . \quad (3.19)$$

The saddle-point equation is

$$\Sigma_0^2 \Lambda - \Lambda^{-1} = 0 . \quad (3.20)$$

The saddle-point equation (3.20) is invariant under $GL(N_{\text{T}}|N_{\text{T}})$ rotations of the supermatrix Λ . We shall therefore solve it first for diagonal $\Lambda = \Lambda_0$, and then determine the adjoint orbit of Λ_0 under the supergroup $GL(N_{\text{T}}|N_{\text{T}})$ which will produce the full solution superspace of (3.20). For this, we set all the Grassmann variables to zero and consider diagonal fermion-fermion and boson-boson blocks (Recall from the previous section that this is in fact all that is required to deduce the pattern of flavour symmetry breaking). The saddle-points are then

$$\Lambda_0 = \frac{1}{\Sigma_0} \hat{\Gamma} \quad (3.21)$$

where $\hat{\Gamma}$ is an $(N_T|N_T)$ diagonal supermatrix with $\hat{\Gamma}^2 = \mathbb{1}_{N_T|N_T}$. In general, the matrix (3.21) does not lie in the integration domain required for uniform convergence of the integration over the boson-boson variables. We will require that, via Cauchy's theorem, the integration contour can be analytically continued into the saddle-point manifold without crossing any of the poles of the function $\text{SDet}^N(\Lambda - i\hat{\mathcal{M}})$. The signs of the eigenvalues of $\bar{\lambda}_0$ are then uniquely determined by the supermatrix Ω , such that the restriction $\Lambda_0^\dagger = \Omega \Lambda_0 \Omega$ is satisfied. In this way we find that analyticity and the forced choice of integration domain for Λ fix the boson-boson part of $\hat{\Gamma}$ to be $\text{diag}(\mathbb{1}_{n_f}, -\mathbb{1}_{n_f}, \mathbb{1}_{n_v^+}, -\mathbb{1}_{n_v^-})$. The fermion-fermion block of $\hat{\Gamma}$, for which there are no convergence nor analyticity constraints, depends on which configuration will dominate the superintegral in the limit $N \rightarrow \infty$. We will now determine this block using supersymmetry.

For this, we expand the function (3.19) about the critical point (3.21) and evaluate the resulting Gaussian fluctuation integral in (3.18). We have

$$\mathcal{F}(\Lambda_0 + Q) = \mathcal{F}(\Lambda_0) + \frac{\Sigma_0^2}{2} \text{STr} \left(Q^2 + \hat{\Gamma} Q \hat{\Gamma} Q \right) + \mathcal{O}(Q^3) \quad (3.22)$$

where $Q \in gl(N_T|N_T)$. Since the supermatrix $\hat{\Gamma}$ defines a projection operator on the linear space $gl(N_T|N_T)$, we can make an orthogonal decomposition $Q = Q_e + Q_o$ (with respect to the inner product $\langle X, Y \rangle = \text{STr} XY$) corresponding to the ± 1 eigenspaces of $\hat{\Gamma}$,

$$\hat{\Gamma} Q_e \hat{\Gamma} = Q_e \quad , \quad \hat{\Gamma} Q_o \hat{\Gamma} = -Q_o . \quad (3.23)$$

Then the quadratic fluctuations (3.22) depend only on the even degrees of freedom Q_e ,

$$\mathcal{F}(\Lambda_0 + Q) = \mathcal{F}(\Lambda_0) + \Sigma_0^2 \text{STr} Q_e^2 + \mathcal{O}(Q^3) . \quad (3.24)$$

The invariant Berezin measure factorizes as $D\Lambda = DQ_e DQ_o$. Upon substitution of (3.24) into (3.18), the integration over the Gaussian fluctuations Q_e around the saddle-point produces one factor of N^{-1} (resp. N^{+1}) for each commuting (resp. anticommuting) direction of steepest descent. The limit $N \rightarrow \infty$ of (3.18) is therefore dominated by those extremal hypersurfaces which have maximal transverse super-dimension $d_f^\perp - d_b^\perp$. This dimension is zero when the fermion-fermion and boson-boson blocks of $\hat{\Gamma}$ are identical, and then the Gaussian fluctuation integral over the modes Q_e produces unity in the limit $N \rightarrow \infty$. Therefore, the dominant saddle-point configuration is given by the supermatrix

$$\hat{\Gamma} = \text{diag} \left(\mathbb{1}_{n_f}, -\mathbb{1}_{n_f}, \mathbb{1}_{n_v^+}, -\mathbb{1}_{n_v^-} \mid \mathbb{1}_{n_f}, -\mathbb{1}_{n_f}, \mathbb{1}_{n_v^+}, -\mathbb{1}_{n_v^-} \right) . \quad (3.25)$$

Having integrated out the degrees of freedom Q_e transverse to the saddle-point supermanifold, let us now focus on the degrees of freedom Q_o tangent to it. The stabilizer subgroup of the matrix (3.25), i.e. the group of matrices $U \in GL(N_T|N_T)$ with $U \hat{\Gamma} U^{-1} = \hat{\Gamma}$, is isomorphic to $GL(n_f + n_v^+ | n_f + n_v^+) \times GL(n_f + n_v^- | n_f + n_v^-)$, and so the saddle-point supermanifold, defined by the adjoint $GL(N_T|N_T)$ orbits of (3.21), is the coset superspace

$$\hat{G}(n_f; n_v^+, n_v^-) = \frac{GL(N_T|N_T)}{GL(n_f + n_v^+ | n_f + n_v^+) \times GL(n_f + n_v^- | n_f + n_v^-)} . \quad (3.26)$$

The measure DQ_0 is the local invariant Berezin measure of this space at the point Λ_0 . We now substitute the adjoint orbits $\Lambda = U \Lambda_0 U^{-1}$ into (3.18) and use the fact [20] that the integration measure $D\Lambda$ coincides with the invariant measure for integration over the coset space (3.26). By using $\mathcal{F}(\Lambda_0) = 0$, we then arrive at the expression for the matrix model partition function (3.3) in the microscopic limit which has a hyperbolic supersymmetry,

$$\mathcal{Z}_{N_f, N_v}^{(\infty)}(\{m_i, \bar{m}_i\}; \{\mu_i, \bar{\mu}_i\}) = \int_{\hat{\mathcal{G}}(n_f; n_v^+, n_v^-)} DU \, e^{-i N \Sigma_0 \, \text{STr}(\hat{\mathcal{M}} U \hat{\Gamma} U^{-1})}, \quad (3.27)$$

where here DU denotes a normalized invariant Haar-Berezin measure on the Lie supergroup $GL(N_T|N_T)$.

Finally, to obtain the partition function for the partially quenched QCD₃ partition function in the microscopic domain, we need to take the large mass limit (3.4). This limit is a little subtle and is described in appendix A, where it is shown that (3.27) reduces to

$$Z_{N_f, N_v}^{(\infty)}(\{m_i\}; \{\mu_i, \bar{\mu}_i\}) = \int_{\hat{\mathcal{G}}(n_f; n_v^+, n_v^-)} DU \, e^{-i N \Sigma_0 \, \text{STr}(\hat{\mathcal{M}}' U \hat{\Gamma}' U^{-1})}, \quad (3.28)$$

where

$$\hat{\mathcal{M}}' = \text{diag} \left(m_1, \dots, m_{N_f}, \mu_1, \dots, \mu_{N_v} \mid \bar{\mu}_1, \dots, \bar{\mu}_{N_v} \right) \quad (3.29)$$

is the mass matrix of the quantum field theory (1.12), and

$$\hat{\Gamma}' = \text{diag} \left(\mathbb{1}_{n_f}, -\mathbb{1}_{n_f}, \mathbb{1}_{n_v^+}, -\mathbb{1}_{n_v^-} \mid \mathbb{1}_{n_v^+}, -\mathbb{1}_{n_v^-} \right). \quad (3.30)$$

Heuristically, the reduction of (3.27) to (3.28) in the regime where the modes of the ghost sea quarks are irrelevant can be understood by appealing once again to the Vafa-Witten theorem. The Hubbard-Stratonovich variable Λ which was introduced in (3.13) may be thought of as the corresponding supermatrix of quark condensates, and therefore, under the assumption of spontaneous flavour symmetry breaking, it should have eigenvalues of the same signs as those of the corresponding mass matrix (3.7). Carrying out the colour-flavour transformation directly for the partition function (3.1) itself (see appendix A), a repeat of the saddle-point analysis of this subsection using these symmetry breaking arguments to determine the dominant configurations would lead exactly to the effective field theory (3.28). Indeed, the integration domain (1.16) in (3.28) is the Goldstone manifold for the flavour supersymmetry breaking pattern (1.15), and the finite volume partition function (3.28) is the appropriate generalization of the Itzykson-Zuber type integral (1.6). Note that the integration domain for the low-momentum Goldstone modes fits into the Zirnbauer classification of the local scaling limits of random matrix theories [25]. This Riemannian symmetric superspace is supported by the compact symmetric space $U(N_f + N_v)/U(n_f + n_v^+) \times U(n_f + n_v^-)$ in the fermion-fermion sector, and by the non-compact symmetric space $U(n_v^+, n_v^-)/U(n_v^+) \times U(n_v^-)$ in the boson-boson sector. These integration manifolds are defined by the intersection of the adjoint orbits of Λ_0 above with the forced integration domain for Λ .

4 Quenched Approximation

As a warm-up to the general case, in this section we will study the finite volume partition function in the fully quenched limit $N_f = 0$. Since our ultimate goal is to obtain explicit expressions for the microscopic spectral density ρ_s , we will concentrate on the case of only a single species of valence quarks, $N_v = 1$. This particular case can be worked out in complete detail and it will serve to illustrate some of the general formalism of the previous section. In particular, it will shed light on the role of supersymmetry in the various manipulations. For example, a crucial issue within the present formalism is the supersymmetric limit of the effective field theory (3.28). According to the general setup, in that case the partition function of partially quenched QCD₃ should reduce to that of ordinary QCD₃,

$$Z_{N_f, N_v}^{(\infty)}(\{m_i\}; \{\mu_i, \bar{\mu}_i\}) \Big|_{\{\mu_i = \bar{\mu}_i\}} = Z_{N_f}^{\text{LS}}(\mathcal{M}) . \quad (4.1)$$

This reduction of the supersymmetric integral (3.28) is highly non-trivial and is related to the notorious boundary ambiguities, or Efetov-Wegner terms [27], which plague integrals over super-manifolds. They are related to non-integrable singularities of the Berezin measure at the boundary of the integration domain in certain coordinate parametrizations whereby the volume form of a non-compact supermanifold becomes a form-valued differential operator. Some formal mathematical descriptions of these boundary terms can be found in [25, 28, 29]. In appendix A we demonstrate the reduction (4.1) formally starting from the finite N representation (3.16). In this section we shall see how it comes about through explicit calculations.

We first consider the saddle-point analysis of the previous section in this special instance. There we saw that the dominant configuration in the large N limit is determined by the supermatrix $\hat{\Gamma} = \text{sgn}(\mu) \mathbb{1}_{1|1}$, for which $Q = Q_e$. The main consequence of this result is that the saddle-point supermanifold is a single-point, since $U \Lambda_0 U^{-1} = \Lambda_0$ for all $U \in GL(1|1)$, and the partition function localizes onto its value at the unique critical point $\Lambda_0 = \frac{\text{sgn} \mu}{\Sigma_0} \mathbb{1}_{1|1}$. By substituting this solution into (3.18), we arrive at the final result for the fully quenched QCD₃ finite volume partition function in the static limit,

$$Z_{0,1}^{(\infty)}(\mu, \bar{\mu}) = e^{-i \text{sgn}(\mu) \text{STr } \hat{\mathcal{M}}_s} , \quad (4.2)$$

which coincides with the result (2.12) obtained directly from the quantum field theory. Note that in the degenerate case $\mu = \bar{\mu}$, we have $Z_{0,1}^{(\infty)}(\mu, \mu) = 1$, as expected. These results simply reflect the fact that, in the microscopic regime of fully quenched QCD₃, there is no spontaneous breakdown of the $GL(1|1)$ flavour supersymmetry.

Let us now go back and consider the partition function (3.16) in the quenched limit,

$$Z_{0,1}(\mu, \bar{\mu}) = \int_{gl(1|1)} D\Lambda \, e^{-\frac{N\Sigma_0^2}{2} \text{STr } \Lambda^2} \text{SDet}^N \left[\Lambda - i \begin{pmatrix} \mu & 0 \\ 0 & \bar{\mu} \end{pmatrix} \right] . \quad (4.3)$$

Note that for $\mu = \bar{\mu}$, the integrand of (4.3) is a supersymmetric invariant function, so that the Efetov-Wegner theorem implies $Z_{0,1}(\mu, \mu) = 1$ (see appendix A, eq. (A.10)). On substituting in the parametrization (3.14) and integrating over the Grassmann variables, eq. (4.3) becomes

$$Z_{0,1}(\mu, \bar{\mu}) = \frac{N}{i^N \sqrt{\pi}} \int_{-\infty}^{\infty} d\lambda \int_{-\infty}^{\infty} d\bar{\lambda} e^{-\frac{N\Sigma_0^2}{2}(\lambda^2 + \bar{\lambda}^2)} \left[\frac{(\lambda - i\mu)^{N-1}}{(\bar{\lambda} - \bar{\mu} + i \operatorname{sgn}(\mu) \epsilon)^{N+1}} - i \Sigma_0^2 \frac{(\lambda - i\mu)^N}{(\bar{\lambda} - \bar{\mu} + i \operatorname{sgn}(\mu) \epsilon)^N} \right], \quad (4.4)$$

where the parameter $\epsilon \rightarrow 0^+$ regulates the poles of the integrand at $\bar{\lambda} = \bar{\mu}$. Note that, according to the general analysis of the previous section, the particular choice of analytic continuation of the integration domain into either the upper or lower complex half-plane depends on the sign of the valence masses. This sign dependence is required for convergence of the Hubbard-Stratonovich transformation of section 3.1.

The integrations over λ and $\bar{\lambda}$ in (4.4) decouple. The λ integrals can be evaluated in terms of Hermite polynomials $H_n(x)$ [30] by using the integral representation

$$H_n(x) = \frac{(2i)^n}{\sqrt{\pi}} \int_{-\infty}^{\infty} dt (t - ix)^n e^{-t^2}, \quad n \geq 0. \quad (4.5)$$

For the $\bar{\lambda}$ integrals, it is convenient to rewrite them in terms of shifted Gaussian moment integrals by using the Fourier transformation

$$\frac{1}{(\bar{\lambda} - \bar{\mu} + i \operatorname{sgn}(\mu) \epsilon)^n} = \frac{(i \operatorname{sgn}(\mu))^n}{(n-1)!} \int_0^{\infty} dk k^{n-1} e^{i \operatorname{sgn}(\mu) k (\bar{\lambda} - \bar{\mu} + i \operatorname{sgn}(\mu) \epsilon)} \quad (4.6)$$

which is valid for $n > 0$. By substituting (4.6) into (4.4), and performing the Gaussian integrals over $\bar{\lambda}$, the remaining integrations over k can be expressed in terms of the generalized Hermite functions [31]

$$\mathcal{H}_n(x) = \frac{(-2i)^{n+1}}{\sqrt{\pi}} e^{x^2} \int_0^{\infty} dt t^n e^{-t^2 - 2ixt}, \quad n \geq 0. \quad (4.7)$$

The Hermite functions are non-polynomial and they are related to the error function. Their imaginary parts coincide with the Hermite polynomials (4.5), while their real parts represent the second set of linearly independent solutions of the Hermite differential equation which can be expressed in terms of confluent hypergeometric functions as

$$\operatorname{Re} \mathcal{H}_n(x) = \begin{cases} (-1)^k x {}_1F_1\left(\frac{1}{2} - k; \frac{3}{2}; x^2\right) & , \quad n = 2k \\ (-1)^{k+1} {}_1F_1\left(\frac{1}{2} - k; \frac{1}{2}; x^2\right) & , \quad n = 2k - 1. \end{cases} \quad (4.8)$$

Combining these results, we arrive at the exact expression for the finite volume partition function of fully quenched QCD₃ with one species of valence quarks,

$$\begin{aligned} Z_{0,1}(\mu, \bar{\mu}) &= \frac{\sqrt{\pi}}{2^N (N-1)!} e^{-N\Sigma_0^2 \bar{\mu}^2/2} \left[H_N \left(\sqrt{\frac{N\Sigma_0^2}{2}} \mu \right) \mathcal{H}_{N-1} \left(\sqrt{\frac{N\Sigma_0^2}{2}} |\bar{\mu}| \right) \right. \\ &\quad \left. - H_{N-1} \left(\sqrt{\frac{N\Sigma_0^2}{2}} \mu \right) \mathcal{H}_N \left(\sqrt{\frac{N\Sigma_0^2}{2}} |\bar{\mu}| \right) \right] . \end{aligned} \quad (4.9)$$

Let us now demonstrate that (4.9) leads to the properties of the finite volume partition function deduced by formal arguments above. First of all, in the degenerate limit $\mu = \bar{\mu}$, we may use the generalized Christoffel-Darboux formula to deduce [31]

$$H_n(x) \mathcal{H}_{n-1}(x) - \mathcal{H}_n(x) H_{n-1}(x) = \frac{2^n (n-1)!}{\sqrt{\pi}} e^{x^2} , \quad (4.10)$$

which when applied to (4.9) leads to $Z_{0,1}(\mu, \mu) = 1$, as anticipated. This demonstrates once again that the valence mass independence of the degenerate partition function is a highly non-trivial result of the exotic properties of superintegrals. Secondly, let us check the microscopic limit of the partition function (4.9). For this, we need the asymptotic form of the Hermite functions [31]

$$\mathcal{H}_n(x) \xrightarrow{n \rightarrow \infty} e^{x^2} n^{n/2} e^{-\frac{1}{2}n - \sqrt{2n}ix} , \quad (4.11)$$

and the standard asymptotic forms of the Hermite polynomials [30]

$$H_n(x) \xrightarrow{n \rightarrow \infty} \begin{cases} (-1)^k 2^k (2k-1)!! e^{x^2} \cos(\sqrt{4k+1}x) & , \quad n = 2k \\ -(-1)^k 2^{k-\frac{1}{2}} (2k-2)!! \sqrt{2k-1} e^{x^2} \sin(\sqrt{4k-1}x) & , \quad n = 2k-1 . \end{cases} \quad (4.12)$$

We substitute (4.11) and (4.12) into (4.9) and take the $N \rightarrow \infty$ limit with the rescaled masses $N\Sigma_0 \mu$, $N\Sigma_0 \bar{\mu}$ fixed. By using the Stirling approximations

$$n! \xrightarrow{n \rightarrow \infty} \sqrt{2\pi n} n^n e^{-n} , \quad n!! \xrightarrow{n \rightarrow \infty} \sqrt{2\pi} n^{n/2} e^{-n/2} , \quad (4.13)$$

we then find that (4.9) in the local scaling limit reduces *exactly* to the anticipated result (2.12) for the microscopic partition function in this case.

5 Microscopic Spectral Density

We finally come to the evaluation of the microscopic spectral density $\rho_s(u; \omega_1, \dots, \omega_{N_f})$ from the finite volume, supersymmetric field theories (3.28). We will compare the expressions obtained from this analysis with those computed in [10, 14] using random matrix theory techniques. In the next section we shall generalize this analysis to compute *all*

microscopic k -point spectral correlation functions. This will thereby demonstrate that the supersymmetric formulation of partially quenched effective field theories provides an analytical framework in which one can establish the equivalence between the microscopic Dirac operator spectrum of QCD₃ and the microscopic spectral correlators of random matrix theory. We shall begin with the quenched approximation to QCD₃, and then move on to the general case of $N_f > 0$ flavours of dynamical fermions.

5.1 Quenched Limit

In the quenched approximation $N_f = 0$ the partition function in the microscopic domain is given by (2.12). The valence quark mass dependence of the fermion condensate may be computed by using (1.13) to get

$$\Sigma_s(u) = \Sigma_0 \frac{i u}{|u|} . \quad (5.1)$$

The function (5.1) has a jump discontinuity of $2i \Sigma_0$ across the real axis. By using (1.14) we thereby find that the microscopic spectral density is given by

$$\rho_s(u) = \frac{1}{\pi} , \quad (5.2)$$

which is the expected result in this case [10, 14]. The spectral distribution function (5.2) is flat and it coincides with the usual macroscopic density $\rho(\lambda)$ evaluated at the spectral origin. Of course this result is not unexpected, given the absence of dynamical fermions. The eigenvalue density near the zero mass regime would normally contain an oscillatory fine structure with period set by the mean level spacing $\pi/N\Sigma_0$. In the present case, this fine structure is absent, and the eigenvalues of the fully quenched QCD₃ Dirac operator are on average uniformly distributed over the real line.

5.2 QCD₃ with N_f Flavours

To treat the general case, we need an appropriate parametrization of the Goldstone manifold $\hat{\mathcal{G}}(n_f; 1^{\text{sgn } \mu}, 0)$ in (1.16). The ordinary integration manifold supporting this coset is the symmetric space $U(2n_f + 1)/U(n_f) \times U(n_f + 1)$ in the fermion-fermion sector, while it is simply a point in the boson-boson sector. Unitarity then requires that the $(2n_f + 1)$ -dimensional anticommuting vectors χ and $\bar{\chi}$ which comprise the Grassmann components of the corresponding Goldstone superfields U obey the constraint $\chi \tilde{U}^{-1} \bar{\chi} = 0$, where \tilde{U} are the commuting, unitary fermion-fermion degrees of freedom of U . Because of the asymmetrical decomposition of this vacuum manifold, it is difficult to evaluate the integral (3.28) in a straightforward way.

To circumvent this asymmetry, we proceed in a way that is reminiscent of the observation [14] that the microscopic spectral density is related to the finite volume effective field

theory for QCD₃ involving two additional species of quarks of equal imaginary mass. By using (1.13,1.14) it is straightforward to show that the spectral density in the mesoscopic domain of the underlying finite volume gauge theory may be computed directly from the partially quenched partition function with *two* flavours of valence quarks as [32]

$$\begin{aligned} & \rho_s(u; \omega_1, \dots, \omega_{N_f}) \\ &= \frac{1}{N\Sigma_0} \frac{1}{Z_{N_f,0} \left(\frac{\mathcal{M}_s}{N\Sigma_0} \right)} \lim_{\epsilon \rightarrow 0^+} \frac{\epsilon}{\pi} \frac{\partial}{\partial \mu_1} \frac{\partial}{\partial \mu_2} Z_{N_f,2} \left(\frac{\mathcal{M}_s}{N\Sigma_0}; \mu_1, -\mu_2, \bar{\mu}_1, -\bar{\mu}_2 \right) \bigg|_{\substack{\mu_1 = \bar{\mu}_1 = \frac{i u}{N\Sigma_0} + \epsilon \\ \mu_2 = \bar{\mu}_2 = -\frac{i u}{N\Sigma_0} + \epsilon}}. \end{aligned} \quad (5.3)$$

In (5.3) the masses $\mu_i, \bar{\mu}_i$ are all positive initially and then analytically continued into the right complex half-plane. To prove the identity (5.3), we note that by using (1.12) the right-hand side may be computed to be

$$- \lim_{\epsilon \rightarrow 0^+} \frac{\epsilon}{\pi} \sum_{n,m} \frac{1}{\lambda_m - \lambda_n - 2i\epsilon} \left(\frac{1}{\lambda_n - \lambda + i\epsilon} - \frac{1}{\lambda_m - \lambda - i\epsilon} \right) \quad (5.4)$$

where λ_n are the Euclidean Dirac operator eigenvalues which are assumed to be non-degenerate. Because of the overall factor of ϵ in (5.4), the $m \neq n$ terms each vanish in the limit $\epsilon \rightarrow 0^+$. This same factor is cancelled by each of the $m = n$ terms in (5.4), which when summed reproduce the expression (1.14). Again it is convenient to evaluate the partially quenched partition function in (5.3) by using a completely supersymmetric expression for it. As in section 3, this is achieved by introducing very heavy superpartners to the dynamical fermions with mass matrix (3.2). Given that the large mass and $N \rightarrow \infty$ limits commute (see appendix A), to treat the thermodynamic limit it is convenient to write the large mass expansion in terms of a ratio of finite volume partition functions as

$$Z_{N_f,2}^{(\infty)}(\mathcal{M}; \mu_1, -\mu_2, \bar{\mu}_1, -\bar{\mu}_2) = Z_{N_f,0}^{(\infty)}(\mathcal{M}) \lim_{\bar{\mathcal{M}} \rightarrow \infty} \frac{\mathcal{Z}_{N_f,2}^{(\infty)}(\mathcal{M}, \bar{\mathcal{M}}; \mu_1, -\mu_2, \bar{\mu}_1, -\bar{\mu}_2)}{\mathcal{Z}_{N_f,0}^{(\infty)}(\mathcal{M}, \bar{\mathcal{M}})}. \quad (5.5)$$

The partition functions appearing on the right-hand side of (5.5) can now be readily analysed.

Coset Parametrization

The integral $\mathcal{Z}_{N_f,2}^{(\infty)}$ is given by (3.27) defined over the symmetric superspace $\hat{G}(n_f; 1, 1)$ in (3.26). To parametrize the supermatrices U of this coset space, it is convenient to change basis on the underlying supervector space $\mathbb{C}^{N_T | N_T} = \mathbb{C}^{n_f+1 | n_f+1} \oplus \mathbb{C}^{n_f+1 | n_f+1}$ to an orthogonal decomposition into the ± 1 eigenspaces of the projection operator (3.25). In this basis, the parity matrix $\hat{\Gamma}$ takes the form

$$\hat{\Gamma} = \mathbb{1}_{n_f+1 | n_f+1} \otimes \sigma_3 \quad (5.6)$$

while the mass matrix (3.7) becomes

$$\hat{\mathcal{M}} = \text{diag}(M_1, -M_2) \quad (5.7)$$

with

$$\begin{aligned} M_i &= \text{diag} \left(\zeta_1^{(i)}, \dots, \zeta_{n_f+1}^{(i)} \middle| \bar{\zeta}_1^{(i)}, \dots, \bar{\zeta}_{n_f+1}^{(i)} \right) \\ &\equiv \text{diag} \left(m_1, \dots, m_{n_f}, \mu_i \middle| \bar{m}_1, \dots, \bar{m}_{n_f}, \bar{\mu}_i \right) \end{aligned} \quad (5.8)$$

for $i = 1, 2$. By exponentiating the coset generators it is then straightforward to show that the matrices $U \in \hat{G}(n_f; 1, 1)$ can be parametrized as

$$U = \begin{pmatrix} \sqrt{1 + \Upsilon \bar{\Upsilon}} & \Upsilon \\ \bar{\Upsilon} & \sqrt{1 + \Upsilon \bar{\Upsilon}} \end{pmatrix}, \quad (5.9)$$

where the supermatrices $\Upsilon, \bar{\Upsilon} \in GL(n_f + 1 | n_f + 1)$ are related by

$$\bar{\Upsilon} = \text{diag} \left(\mathbb{1}_{n_f+1} \middle| -\mathbb{1}_{n_f+1} \right) \Upsilon^\dagger. \quad (5.10)$$

In these Cartesian coordinates, the invariant measure for integration over the coset space is given by

$$DU = \prod_{i,j=1}^{n_f+1} d\Upsilon_{ij} d\Upsilon_{ij}^* \prod_{\alpha,\beta=1}^{n_f+1} d\Upsilon_{\alpha\beta} d\Upsilon_{\alpha\beta}^* \otimes \prod_{k=1}^{n_f+1} \prod_{\sigma=1}^{n_f+1} \frac{\partial}{\partial \Upsilon_{k\sigma}} \frac{\partial}{\partial \Upsilon_{k\sigma}^*} \frac{\partial}{\partial \Upsilon_{\sigma k}} \frac{\partial}{\partial \Upsilon_{\sigma k}^*}. \quad (5.11)$$

However, despite the simplicity of the integration measure (5.11), Cartesian coordinates are not convenient for the evaluation of the integrals (3.27). Following [29, 33], we introduce the Efetov polar coordinate parametrization of the coset space [34]. These coordinates are inherited from the decomposition of $U(N_T | N_T)$ matrices into eigenvalue and angular degrees of freedom. When projected onto the coset, these variables form an orthogonal decomposition into parity even and odd sectors. Namely, we may parametrize the elements of the coset space as

$$U = V \Xi V^{-1}, \quad (5.12)$$

where the anticommuting coordinates reside in the angular, “eigenvector” matrix V which commutes with the coset projection operator,

$$V \hat{\Gamma} = \hat{\Gamma} V, \quad (5.13)$$

while the commuting coordinates live in the “eigenvalue” matrix Ξ which anticommutes with the parity matrix,

$$\Xi \hat{\Gamma} = \hat{\Gamma} \Xi^{-1}. \quad (5.14)$$

From (5.13) it follows that the angular matrices $V \in U(n_f + 1 | n_f + 1) \times U(n_f + 1 | n_f + 1) / U(1)^{n_f+1 | n_f+1}$ admit the matrix presentation

$$V = \text{diag}(V_+, V_-) \quad (5.15)$$

in the parity ordered basis introduced above. The matrices Ξ satisfying (5.14) may be parametrized as

$$\Xi = \begin{pmatrix} \sqrt{\frac{\mathcal{R}+1}{2}} & \sqrt{\frac{\mathcal{R}-1}{2}} \\ \sqrt{\frac{\mathcal{R}-1}{2}} & \sqrt{\frac{\mathcal{R}+1}{2}} \end{pmatrix}, \quad (5.16)$$

where

$$\mathcal{R} = \text{diag} \left(r_1, \dots, r_{n_f+1} \middle| \bar{r}_1, \dots, \bar{r}_{n_f+1} \right). \quad (5.17)$$

The compact fermion-fermion radial coordinates r_i each live in the finite interval $[-1, 1]$, while the non-compact boson-boson radial coordinates \bar{r}_α live in the semi-infinite interval $[1, \infty)$. The collection of matrices (5.16, 5.17) form an $(n_f + 1 | n_f + 1)$ maximal abelian subgroup for the Cartan decomposition of $GL(N_T | N_T)$ with respect to the stability subgroup $GL(n_f + 1 | n_f + 1) \times GL(n_f + 1 | n_f + 1)$. We recall from section 3.2 that the latter degrees of freedom defined the directions of steepest descent on the saddle point manifold, while the former ones determined its structure.

There are two main advantages of this polar coordinate parametrization. First of all, while the original integration in (3.27) cannot be trivially extended from the Goldstone supermanifold to the full flavour supergroup, the angular integrations over V_\pm can be extended to the whole unitary supergroup $U(n_f + 1 | n_f + 1)$. Then, the appropriate supersymmetric generalization of the Itzykson-Zuber formula may be applied. Secondly, the choice of coordinates (5.12) will completely decouple the bosonic and fermionic sectors of the supergroup integral from one another in such a way that the large mass limit (5.5) may be easily taken. The price to pay for the introduction of these coordinates is that the radial integration domain has a boundary, and so we can anticipate the appearance of Efetov-Wegner terms. The calculation of the Berezinian of the coordinate transformation (5.12) can be done in the usual way [21] and the measure assumes the familiar form

$$DU = C_0 \, DV_+ \, DV_- \prod_{i=1}^{n_f+1} dr_i \prod_{\alpha=1}^{n_f+1} d\bar{r}_\alpha \frac{\Delta[r]^2 \Delta[\bar{r}]^2}{\Delta[r, \bar{r}]^2}, \quad (5.18)$$

where DV_\pm are invariant Haar-Berezin measures on $U(n_f + 1 | n_f + 1)$ and the Δ 's are $(n_f + 1) \times (n_f + 1)$ Vandermonde determinants which are defined in appendix B. Here and in the following we will, for simplicity, not keep track of numerical integration factors and simply denote them collectively by C_0 . The appropriate normalization will be restored by hand in our final result later on. From (5.18) we see that the Berezin measure in polar coordinates contains non-integrable singularities at the points $r_i = \bar{r}_\alpha = 1$ for any pair (i, α) , unlike the analytic Cartesian coordinate measure (5.11). These fictitious singularities are caused by the mixing of nilpotent terms into the commuting degrees of freedom in (5.12) which lead to total derivatives that give rise to additional contributions to the pertinent integral at the boundaries of the radial integration domain. Although these boundary terms can be calculated in principle by using the techniques described in [29], they are rather cumbersome in form and not very informative. In what follows we

will show that, as in [33], they do not contribute in the limits of interest. Heuristically, these terms arise from the introduction of the fictitious superpartners to the physical quarks and the valence fermions. They therefore should not contribute to any physical observable, such as the Dirac operator spectrum.

Finite Volume Partition Functions

We will now evaluate the partition function $\mathcal{Z}_{N_f,2}^{(\infty)}$ by simply ignoring the Efetov-Wegner terms. We substitute (5.12), (5.15), (5.16), and (5.18) into (3.27). By using the commutation relations (5.13) and (5.14) we find that the supertrace factorizes into parity sectors as

$$\text{STr} \left(\hat{\mathcal{M}} V \Xi V^{-1} \hat{\Gamma} V \Xi^{-1} V^{-1} \right) = \text{STr} \left(M_1 V_+ \mathcal{R} V_+^{-1} \right) + \text{STr} \left(M_2 V_- \mathcal{R} V_-^{-1} \right), \quad (5.19)$$

and so the partition function can be expressed in a factorized form as

$$\begin{aligned} \mathcal{Z}_{N_f,2}^{(\infty)} \left(\mathcal{M}, \bar{\mathcal{M}}; \mu_1, -\mu_2, \bar{\mu}_1, -\bar{\mu}_2 \right) &= C_0 \prod_{i=1}^{n_f+1} \int_{-1}^1 dr_i \prod_{\alpha=1}^{n_f+1} \int_1^\infty d\bar{r}_\alpha \frac{\Delta[r]^2 \Delta[\bar{r}]^2}{\Delta[r, \bar{r}]^2} \\ &\times \int_{U(n_f+1|n_f+1)} DV_+ e^{-i N \Sigma_0 \text{STr} (M_1 V_+ \mathcal{R} V_+^{-1})} \\ &\times \int_{U(n_f+1|n_f+1)} DV_- e^{-i N \Sigma_0 \text{STr} (M_2 V_- \mathcal{R} V_-^{-1})}. \end{aligned} \quad (5.20)$$

The unitary integrals in (5.20) can each be evaluated by using the supersymmetric generalization of the Itzykson-Zuber formula [22] (see appendix B, eq. (B.13)) to get

$$\begin{aligned} \mathcal{Z}_{N_f,2}^{(\infty)} \left(\mathcal{M}, \bar{\mathcal{M}}; \mu_1, -\mu_2, \bar{\mu}_1, -\bar{\mu}_2 \right) &= C_0 \prod_{k=1,2} \frac{\Delta \left[\zeta^{(k)}, \bar{\zeta}^{(k)} \right]}{\Delta \left[\zeta^{(k)} \right] \Delta \left[\bar{\zeta}^{(k)} \right]} \\ &\times \prod_{i=1}^{n_f+1} \int_{-1}^1 dr_i \prod_{\alpha=1}^{n_f+1} \int_1^\infty d\bar{r}_\alpha \prod_{k'=1,2} \det_{i,j} \left[e^{-i N \Sigma_0 \zeta_i^{(k')} r_j} \right] \det_{\alpha,\beta} \left[e^{i N \Sigma_0 \bar{\zeta}_\alpha^{(k')} \bar{r}_\beta} \right]. \end{aligned} \quad (5.21)$$

The radial integrals over r_i and \bar{r}_α in (5.21) decouple. Because of the permutation symmetry of the integration measure, the two fermion-fermion determinants may be combined into a single one $\det_{i,j} \left[e^{-i N \Sigma_0 (\zeta_i^{(1)} + \zeta_j^{(2)}) r_j} \right]$, times the order $(n_f + 1)!$ of the permutation group S_{n_f+1} which we absorb as always into the normalization constant C_0 . The same is true of the boson-boson determinants. The radial integrals may now be straightforwardly done, and by using the definitions of $\zeta_i^{(k)}$ and $\bar{\zeta}_\alpha^{(k)}$ in (5.8) we arrive at

$$\mathcal{Z}_{N_f,2}^{(\infty)} \left(\mathcal{M}, \bar{\mathcal{M}}; \mu_1, -\mu_2, \bar{\mu}_1, -\bar{\mu}_2 \right)$$

$$\begin{aligned}
&= C_0 (\mu_1 - \bar{\mu}_1)(\mu_2 - \bar{\mu}_2) \prod_{i=1}^{n_f} \frac{(m_i - \bar{\mu}_1)(m_i - \bar{\mu}_2)}{(m_i - \mu_1)(m_i - \mu_2)} \prod_{\alpha=1}^{n_f} \frac{(\bar{m}_\alpha - \mu_1)(\bar{m}_\alpha - \mu_2)}{(\bar{m}_\alpha - \bar{\mu}_1)(\bar{m}_\alpha - \bar{\mu}_2)} \\
&\times \frac{\Delta[m, \bar{m}]^2}{\Delta[m]^2 \Delta[\bar{m}]^2} \det \begin{bmatrix} \frac{\sin N\Sigma_0(m_i + m_j)}{N\Sigma_0(m_i + m_j)} & \frac{\sin N\Sigma_0(m_i + \mu_1)}{N\Sigma_0(m_i + \mu_1)} \\ \frac{\sin N\Sigma_0(\mu_2 + m_j)}{N\Sigma_0(\mu_2 + m_j)} & \frac{\sin N\Sigma_0(\mu_1 + \mu_2)}{N\Sigma_0(\mu_1 + \mu_2)} \end{bmatrix} \\
&\times \det \begin{bmatrix} \frac{e^{iN\Sigma_0(\bar{m}_\alpha + \bar{m}_\beta)}}{N\Sigma_0(\bar{m}_\alpha + \bar{m}_\beta)} & \frac{e^{iN\Sigma_0(\bar{m}_\alpha + \bar{\mu}_1)}}{N\Sigma_0(\bar{m}_\alpha + \bar{\mu}_1)} \\ \frac{e^{iN\Sigma_0(\bar{\mu}_2 + \bar{m}_\beta)}}{N\Sigma_0(\bar{\mu}_2 + \bar{m}_\beta)} & \frac{e^{iN\Sigma_0(\bar{\mu}_1 + \bar{\mu}_2)}}{N\Sigma_0(\bar{\mu}_1 + \bar{\mu}_2)} \end{bmatrix}. \tag{5.22}
\end{aligned}$$

In (5.22) the Δ 's denote $n_f \times n_f$ Vandermonde determinants in the fermion masses m_i and \bar{m}_α , $i, \alpha = 1, \dots, n_f$, while the ordinary determinants have dimension $(n_f + 1) \times (n_f + 1)$.

Let us now evaluate the partition function $\mathcal{Z}_{N_f,0}^{(\infty)}$ which appears in the denominator of the expression (5.5). Applying the exact same steps which led to (5.22) for the coset integral (3.27) over $\hat{G}(n_f; 0, 0)$, we arrive at

$$\begin{aligned}
\mathcal{Z}_{N_f,0}^{(\infty)}(\mathcal{M}, \bar{\mathcal{M}}) &= C_0 \frac{\Delta[m, \bar{m}]^2}{\Delta[m]^2 \Delta[\bar{m}]^2} \det_{1 \leq i, j \leq n_f} \left[\frac{\sin N\Sigma_0(m_i + m_j)}{N\Sigma_0(m_i + m_j)} \right] \\
&\times \det_{1 \leq \alpha, \beta \leq n_f} \left[\frac{e^{iN\Sigma_0(\bar{m}_\alpha + \bar{m}_\beta)}}{N\Sigma_0(\bar{m}_\alpha + \bar{m}_\beta)} \right]. \tag{5.23}
\end{aligned}$$

The terms in (5.23) which depend only on the physical fermion masses m_i are readily recognized as the finite volume partition function $Z_{N_f,0}^{(\infty)}(\mathcal{M})$ for ordinary QCD₃. Indeed, one can parametrize the ordinary symmetric space (1.7) in the same manner described above and evaluate the finite volume partition function (1.6) analogously as an integral over the *coset* space, rather than the full unitary flavour symmetry group, by using the ordinary Itzykson-Zuber formula for $U(n_f)$ [35]. In this way one may arrive at the representation (up to an irrelevant normalization factor)

$$Z_{N_f}^{\text{LS}}(\omega_1, \dots, \omega_{N_f}) = \frac{1}{\Delta[\omega]^2} \det_{1 \leq i, j \leq n_f} \left[\frac{\sinh(\omega_i + \omega_j)}{\omega_i + \omega_j} \right], \tag{5.24}$$

where here we have analytically continued the unfolded masses (1.8) to imaginary values to facilitate comparison with previous results. In [10] an expression for the finite volume QCD₃ partition function was derived by applying the ordinary Itzykson-Zuber formula for $U(2n_f)$ in a suitable regulated limit that removes the n_f -fold degeneracy of the eigenvalues of the matrix Γ_5 . The expression (5.24), which utilizes the same integration formula but does not require dealing with any degeneracies, is much simpler and compact as it involves elementary $n_f \times n_f$ determinants, rather than the $2n_f \times 2n_f$ determinants that appear in [10]. While we have no direct proof at present that these two expressions are equivalent, we have checked that they agree in a number of cases. The results (5.23) and (5.24) clearly show that the Efetov-Wegner boundary terms vanish in the decoupling limits $\bar{m}_\alpha \rightarrow \infty$, as

expected. For example, one of these boundary terms comes from evaluating the integrand of (3.27) at the origin $U = \mathbb{1}_{N_T|N_T}$ of the coset superspace [29]. This adds the term $e^{-iN\Sigma_0 \text{STr}(M_1+M_2)}$ to the above expressions, which produces a vanishing result in the large mass limit.

We can now finally write down the desired expression for the partially quenched finite volume partition function. By substituting (5.22)–(5.24) into (5.5) and taking the limit $\bar{m}_\alpha \rightarrow \infty$, we arrive at

$$\begin{aligned}
& Z_{N_f,2}^{(\infty)}(\mathcal{M}; \mu_1, -\mu_2, \bar{\mu}_1, -\bar{\mu}_2) \\
&= C_0 \frac{(\mu_1 - \bar{\mu}_1)(\mu_2 - \bar{\mu}_2)}{\Delta[m]^2} \frac{e^{iN\Sigma_0(\bar{\mu}_1+\bar{\mu}_2)}}{N\Sigma_0(\bar{\mu}_1 + \bar{\mu}_2)} \prod_{i=1}^{n_f} \frac{(m_i - \bar{\mu}_1)(m_i - \bar{\mu}_2)}{(m_i - \mu_1)(m_i - \mu_2)} \\
&\quad \times \det \begin{bmatrix} \frac{\sin N\Sigma_0(m_i + m_j)}{N\Sigma_0(m_i + m_j)} & \frac{\sin N\Sigma_0(m_i + \mu_1)}{N\Sigma_0(m_i + \mu_1)} \\ \frac{\sin N\Sigma_0(\mu_2 + m_j)}{N\Sigma_0(\mu_2 + m_j)} & \frac{\sin N\Sigma_0(\mu_1 + \mu_2)}{N\Sigma_0(\mu_1 + \mu_2)} \end{bmatrix}. \tag{5.25}
\end{aligned}$$

The expression (5.25) of course represents only the bulk, regular contribution to the supersymmetric integral (3.27). The anomalous boundary terms are obtained by setting $r_i = \bar{r}_\alpha = 1$ for one or several pairs of radial coordinates (r_i, \bar{r}_α) . In the original integral (3.27), this corresponds to setting some of the supersymmetric blocks of the unitary matrix U equal to the identity matrix of the appropriate dimensionality. The integrand of (3.27) then becomes a supersymmetric invariant function of the given block, and the integral becomes correspondingly dimensionally reduced (see appendix A). In the integration measure (5.18) the Vandermonde determinants are reduced accordingly by omitting the given singular factors. The reduced coset integral can thereby be evaluated as above. As we have shown, only the boundary terms which are associated with the valence fermions contribute. These terms can be computed using the results of [29]. One of them comes from setting the (2|2) valence block of U equal to the identity matrix. Following the derivations above, this produces the boundary term

$$\begin{aligned}
Z_{N_f,2}^{\text{EW}}(\mathcal{M}; \mu_1, -\mu_2, \bar{\mu}_1, -\bar{\mu}_2) &= \frac{1}{\Delta[m]^2} e^{-iN\Sigma_0(\mu_1-\bar{\mu}_1)} e^{iN\Sigma_0(\mu_2-\bar{\mu}_2)} \\
&\quad \times \det_{1 \leq i,j \leq n_f} \left[\frac{\sin N\Sigma_0(m_i + m_j)}{N\Sigma_0(m_i + m_j)} \right] \tag{5.26}
\end{aligned}$$

which is responsible for the normalization (4.1).

Spectral Distribution Function

Finally, we can now easily compute the spectral density of the QCD₃ Dirac operator in the mesoscopic region using (5.3). We note first of all that only the regular part (5.25) contributes to the spectral density. The factor of $(\bar{\mu}_1 + \bar{\mu}_2)^{-1}$ that appears there is needed to cancel the factor of ϵ in (5.3) in the limit $\bar{\mu}_1 + \bar{\mu}_2 \rightarrow 0$. The boundary terms contain

fewer non-compact integrations, so they either vanish in the limit $\bar{m}_\alpha \rightarrow \infty$, or they do not contain this factor and so vanish in the limit $\epsilon \rightarrow 0^+$. With this in mind, we can now differentiate the expression (5.25) with respect to the valence quark masses μ_i and take the limit dictated in (5.3). By using (5.24), we then arrive at a relatively simple expression for the microscopic spectral density,

$$\rho_s(u; \omega_1, \dots, \omega_{N_f}) = \frac{1}{\pi} \frac{\det \begin{bmatrix} \frac{\sinh(\omega_i + \omega_j)}{\omega_i + \omega_j} & \frac{\sinh(\omega_i + iu)}{\omega_i + iu} \\ \frac{\sinh(\omega_j - iu)}{\omega_j - iu} & 1 \end{bmatrix}}{\det_{1 \leq i, j \leq n_f} \left[\frac{\sinh(\omega_i + \omega_j)}{\omega_i + \omega_j} \right]}, \quad (5.27)$$

where we have again analytically continued both the unfolded masses ω_i and the unfolded Dirac operator eigenvalues u to imaginary values, as these are the standard conventions that are used in gauge theory computations. The determinant in the numerator of (5.27) is of dimension $(n_f + 1) \times (n_f + 1)$, and the overall normalization constant $C_0 = 1/\pi$ has been fixed by the usual matching condition between the microscopic density and the macroscopic density at the spectral origin,

$$\lim_{u \rightarrow \infty} \rho_s(u; \omega_1, \dots, \omega_{N_f}) = \frac{\rho(0)}{N \Sigma_0} = \frac{1}{\pi}. \quad (5.28)$$

5.3 Examples

The universal double-microscopic spectral density was computed in [10] using random matrix theory techniques and found to be given by a rather involved determinant formula involving Bessel functions of half-integer order. Here we have found that the supersymmetric method based entirely on the field theory formulation leads to an elegant expression (5.27) for the same quantity which is much more compact and convenient to use. Again, we have no direct proof of the equivalence of these two representations of the spectral density, but we note that they both involve determinants of the same dimension $(n_f + 1) \times (n_f + 1)$. We have checked that they agree in a number of special cases. For example, consider the case of two physical quarks, $n_f = 1$, of equal and opposite mass m . By using the trigonometric identity

$$2 \sinh(x + y) \sinh(x - y) = \cosh 2x - \cosh 2y \quad (5.29)$$

the resulting 2×2 determinant in (5.27) can be worked out to give the result

$$\rho_s(u; \omega, -\omega) = \frac{1}{\pi} \left(1 + \frac{\omega}{u^2 + \omega^2} \frac{\cos 2u - \cosh 2\omega}{\sinh 2\omega} \right) \quad (5.30)$$

which agrees with the known density of states from random matrix theory [10]. Similar computations can be done for higher numbers of massive fermion flavours, and in each case we have found precise agreement with the results of [10].

An important special case that can be worked out straightforwardly is that of an arbitrary number $N_f = 2n_f$ of *massless* quarks. In this case, the ratio of determinants in (5.27) produces an indeterminate form and must be defined by an appropriate regularization in the limit $\omega_i \rightarrow 0$. In that limit, the argument of the determinant in (5.24) admits the Taylor series expansion

$$f(\omega_i + \omega_j) \equiv \frac{\sinh(\omega_i + \omega_j)}{\omega_i + \omega_j} = \sum_{k,l=1}^{n_f} \mathcal{A}_{kl} \mathcal{B}_{li} \mathcal{B}_{kj} + \mathcal{O}(\omega_i^{n_f}) , \quad (5.31)$$

where we have defined the $n_f \times n_f$ matrices

$$\mathcal{A}_{ij} = f^{(i+j-2)}(0) \quad , \quad \mathcal{B}_{ij} = \frac{\omega_i^{j-1}}{(i-1)!} . \quad (5.32)$$

In writing (5.31) we have used the fact that both the numerator and denominator of (5.24) vanish as $\omega_i^{n_f-1}$ in the limit $\omega_i \rightarrow 0$. Since $\det \mathcal{B} = \Delta[\omega] / \prod_{i=1}^{n_f} (i-1)!$, this regulates the partition function (5.24) and yields the finite result $Z_{N_f}^{\text{LS}}(0, \dots, 0) = \det \mathcal{A} / (\prod_{i=1}^{n_f} (i-1)!)^2$.

By substituting (5.31) into (5.27), the microscopic spectral density can be written as the $(n_f + 1) \times (n_f + 1)$ determinant

$$\rho_s(u; 0, \dots, 0) = \frac{1}{\pi} \det \begin{pmatrix} \mathbb{1}_{n_f} & \mathcal{B}^{-1} \mathcal{A} \vec{a} \\ \vec{a}^\top \mathcal{B}^\top & 1 \end{pmatrix} \quad (5.33)$$

where we have defined the n_f -dimensional vector

$$\vec{a}_j = f^{(j-1)}(iu) . \quad (5.34)$$

The determinant (5.33) is readily evaluated by performing minor expansions along the last row and column with the finite result

$$\rho_s(u; 0, \dots, 0) = \frac{1}{\pi} \left(1 - \vec{a}^\top \mathcal{A} \vec{a} \right) . \quad (5.35)$$

By substituting (5.32) and (5.34) into (5.35), and by using $f^{(2n-1)}(0) = 0$ and $f^{(2n)}(0) = 1/(2n+1)$, the spectral density in the massless limit can thereby be written as

$$\rho_s(u; 0, \dots, 0) = \frac{1}{\pi} \left[1 - \sum_{l=0}^{n_f-1} \frac{(-1)^l}{2l+1} \sum_{k=1}^{2l+1} \left(\frac{d^{k-1}}{du^{k-1}} \frac{\sin u}{u} \right) \left(\frac{d^{2l-k+1}}{du^{2l-k+1}} \frac{\sin u}{u} \right) \right] . \quad (5.36)$$

The function (5.36) can be expressed in terms of regular Bessel functions $J_\nu(x)$ of half-integer order ν [30] by using the derivative formula

$$J_{n+\frac{1}{2}}(x) = (-2)^n x^{n+\frac{1}{2}} \sqrt{\frac{2}{\pi}} \left(\frac{d}{dx^2} \right)^n \frac{\sin x}{x} \quad (5.37)$$

which is valid for positive integral n , and the three-term recursion relations

$$\begin{aligned} 2J'_\nu(x) &= J_{\nu-1}(x) - J_{\nu+1}(x) , \\ \frac{2\nu}{x} J_\nu(x) &= J_{\nu-1}(x) + J_{\nu+1}(x) . \end{aligned} \quad (5.38)$$

After some algebra, the expression (5.36) can be simplified to the compact form

$$\rho_s(u; 0, \dots, 0) = \frac{u}{4} \left[J_{n_f - \frac{1}{2}}(u)^2 - J_{n_f + \frac{1}{2}}(u) J_{n_f - \frac{3}{2}}(u) + J_{n_f + \frac{1}{2}}(u)^2 - J_{n_f - \frac{1}{2}}(u) J_{n_f + \frac{3}{2}}(u) \right] \quad (5.39)$$

which coincides with the massless spectral density obtained originally from random matrix theory [9, 36]. Here we have derived it directly from the effective finite volume partition function of QCD₃ in the microscopic scaling regime.

In [14] it was shown, by matching exact results from random matrix theory with the low energy effective field theory (1.6), that it is possible to express the spectral density of the QCD₃ Dirac operator in terms of a ratio of two finite volume partition functions, one of which involves two additional fermion species of equal imaginary mass, as

$$\rho_s(u; \omega_1, \dots, \omega_{N_f}) = \frac{1}{2\pi} \prod_{i=1}^{n_f} (u^2 + \omega_i^2) \frac{Z_{N_f+2}^{\text{LS}}(\omega_1, \dots, \omega_{N_f}, iu, iu)}{Z_{N_f}^{\text{LS}}(\omega_1, \dots, \omega_{N_f})}. \quad (5.40)$$

As mentioned at the beginning of section 5.2, this is not unexpected as the spectral density is given directly from (5.3). The partition function in the numerator of (5.40) is understood as an analytical continuation in the additional fictitious fermion masses in which both mass parities are substituted by the value iu . The expression (5.27) can thereby be thought of as an explicit realization of this feature, with the appropriate analytic continuation of the coset representation (5.24) given by the numerator function in (5.40). This is the biggest advantage of the polar coordinate parametrization of the coset. The supersymmetric partners to both the sea and valence quarks are just spectators in this formalism, and it is the valence fermions themselves which give the explicit representation of the microscopic Dirac operator spectrum in terms of the effective field theory that is extended by additional fermionic species. Moreover, this result is completely independent of any random matrix theory representation of the quantum field theory.

As noted in [14], the relationship (5.40) gives a much more compact form for the spectral density than that found in [10]. Here we have found an even more convenient expression for it, based on a coset parametrization of the finite volume gauge theory partition function. We stress once more that in the ordinary, unquenched case there is no need for this coset analysis because the group integral extends over $U(N_f)$ up to a unitary group volume factor. However, in the supersymmetric case we are forced to deal directly with the coset space because this volume factor vanishes [21]. The results of section complete the goal that was set out in section 1 of this paper, namely an analytical derivation of the QCD₃ Dirac operator spectrum directly from quantum field theory. As a bonus, the supersymmetric form of the finite volume partially quenched partition function has thereby provided a new and much simpler expression for the microscopic spectral density. This illustrates the power behind the supersymmetric method.

6 Higher Order Spectral Correlation Functions

Just as the equivalence between the universal random matrix theory and low energy effective field theory partition functions is not sufficient to establish the computability of the microscopic Dirac operator spectrum in random matrix theory, neither is merely the computation of the spectral one-point function. To complete the proof, one needs to extend the calculations to derive the generic k -point spectral functions $\rho_s(u_1, \dots, u_k; \{\omega_i\})$. It is clear that in this case one needs to consider a partially quenched quantum field theory involving $N_v \geq k$ species of valence quarks, in which case the higher k -point spectral correlators may be computed as the discontinuity across the cut of a k -th order fermion susceptibility as

$$\rho_s(u_1, \dots, u_k; \{\omega_i\}) = \left(\frac{1}{2\pi i \Sigma_0}\right)^k \lim_{\epsilon \rightarrow 0^+} \prod_{l=1}^k \sum_{\sigma_l = \pm 1} \sigma_l \Sigma_s(u_1 + i\sigma_1 \epsilon, \dots, u_k + i\sigma_k \epsilon; \{\omega_i\}) \quad (6.1)$$

where

$$\begin{aligned} \Sigma_s(i\mu_1, \dots, i\mu_k; \{\omega_i\}) &= \left(-\frac{i}{N}\right)^k \frac{1}{Z_{N_f, 0}\left(\frac{\mathcal{M}_s}{N\Sigma_0}\right)} \\ &\times \prod_{j=1}^k \frac{\partial}{\partial \mu_j} Z_{N_f, N_v}\left(\left\{\frac{\omega_i}{N\Sigma_0}\right\}; \{\mu_i, \bar{\mu}_i\}\right) \Big|_{\{\mu_i = \bar{\mu}_i\}}. \end{aligned} \quad (6.2)$$

In this section we will point out that the method described in the previous section can be used to derive the function (6.1) from a partially quenched field theory partition function involving $N_v = 2k$ species of valence quarks and supersymmetric coset $\hat{G}(n_f; k, k)$ in (3.26). Thus in three spacetime dimensions, the program for computing the full microscopic spectrum of $i\mathcal{D}$ may be completed in a straightforward way, unlike the four dimensional case.

The calculation proceeds in exactly the same manner as in the previous section and we will therefore be very brief, only highlighting the salient points. The spectral k -point function (6.1) may be alternatively derived from [32]

$$\begin{aligned} \rho_s(u_1, \dots, u_k; \{\omega_i\}) &= \left(\frac{1}{N\Sigma_0}\right)^k \frac{1}{Z_{N_f, 0}\left(\frac{\mathcal{M}_s}{N\Sigma_0}\right)} \lim_{\epsilon \rightarrow 0^+} \left(\frac{\epsilon}{\pi}\right)^k \\ &\times \prod_{j=1}^{2k} \frac{\partial}{\partial \mu_j} Z_{N_f, 2k}\left(\frac{\mathcal{M}_s}{N\Sigma_0}; \{\mu_{2l-1}, -\mu_{2l}, \bar{\mu}_{2l-1}, -\bar{\mu}_{2l}\}\right) \Bigg|_{\substack{\mu_{2l-1} = \bar{\mu}_{2l-1} = \frac{i u_l}{N\Sigma_0} + \epsilon \\ \mu_{2l} = \bar{\mu}_{2l} = -\frac{i u_l}{N\Sigma_0} + \epsilon}}. \end{aligned} \quad (6.3)$$

The partially quenched field theory partition function in (6.3) may be evaluated in exactly the same way as described in section 5. For the regular part we find

$$Z_{N_f, 2k}^{(\infty)}(\mathcal{M}; \{\mu_{2l-1}, -\mu_{2l}, \bar{\mu}_{2l-1}, -\bar{\mu}_{2l}\})$$

$$\begin{aligned}
&= \frac{C_0}{\Delta[m]^2} \frac{\prod_{l,l'=1}^k (\mu_{2l-1} - \bar{\mu}_{2l'-1}) (\mu_{2l} - \bar{\mu}_{2l'})}{\prod_{l < l'} (\mu_{2l-1} - \mu_{2l'-1}) (\mu_{2l} - \mu_{2l'}) (\bar{\mu}_{2l-1} - \bar{\mu}_{2l'-1}) (\bar{\mu}_{2l} - \bar{\mu}_{2l'})} \\
&\times \prod_{i=1}^{n_f} \prod_{l=1}^k \frac{(m_i - \bar{\mu}_{2l-1})(m_i - \bar{\mu}_{2l})}{(m_i - \mu_{2l-1})(m_i - \mu_{2l})} \det_{1 \leq l, l' \leq k} \left[\frac{e^{i N \Sigma_0 (\bar{\mu}_{2l-1} + \bar{\mu}_{2l'})}}{N \Sigma_0 (\bar{\mu}_{2l-1} + \bar{\mu}_{2l'})} \right] \\
&\times \det \begin{bmatrix} \frac{\sin N \Sigma_0 (m_i + m_j)}{N \Sigma_0 (m_i + m_j)} & \frac{\sin N \Sigma_0 (m_i + \mu_{2l'})}{N \Sigma_0 (m_i + \mu_{2l'})} \\ \frac{\sin N \Sigma_0 (\mu_{2l-1} + m_j)}{N \Sigma_0 (\mu_{2l-1} + m_j)} & \frac{\sin N \Sigma_0 (\mu_{2l-1} + \mu_{2l'})}{N \Sigma_0 (\mu_{2l-1} + \mu_{2l'})} \end{bmatrix} \quad (6.4)
\end{aligned}$$

where the second determinant is of dimension $(n_f + k) \times (n_f + k)$, while for the Efetov-Wegner term which yields the normalization (4.1) we find

$$\begin{aligned}
&Z_{N_f, 2k}^{\text{EW}}(\mathcal{M}; \{\mu_{2l-1}, -\mu_{2l}, \bar{\mu}_{2l-1}, -\bar{\mu}_{2l}\}) \\
&= \frac{1}{\Delta[m]^2} \prod_{l=1}^k e^{-i N \Sigma_0 (\mu_{2l-1} - \bar{\mu}_{2l-1})} e^{i N \Sigma_0 (\mu_{2l} - \bar{\mu}_{2l})} \det_{1 \leq i, j \leq n_f} \left[\frac{\sin N \Sigma_0 (m_i + m_j)}{N \Sigma_0 (m_i + m_j)} \right]. \quad (6.5)
\end{aligned}$$

The microscopic spectral k -point function now follows from differentiating (6.4) with respect to the valence quark masses as prescribed by (6.3). Note that if we expand the $k \times k$ determinant in (6.4) as a sum over elements of the symmetric group S_{2k} , then only the contribution from the identity permutation survives in the limit $\bar{\mu}_{2l-1} + \bar{\mu}_{2l} \rightarrow 0$, since it is only that term which cancels the factor of ϵ^k in (6.3). In this way we arrive at

$$\rho_s(u_1, \dots, u_k; \{\omega_i\}) = \left(\frac{1}{\pi}\right)^k \frac{\det \begin{bmatrix} \frac{\sinh(\omega_i + \omega_j)}{\omega_i + \omega_j} & \frac{\sinh(\omega_i + i u_{l'})}{\omega_i + i u_{l'}} \\ \frac{\sinh(\omega_j - i u_l)}{\omega_j - i u_l} & \frac{\sinh(u_l - u_{l'})}{u_l - u_{l'}} \end{bmatrix}}{\det_{1 \leq i, j \leq n_f} \left[\frac{\sinh(\omega_i + \omega_j)}{\omega_i + \omega_j} \right]}, \quad (6.6)$$

where the determinant in the numerator of (6.6) is of dimension $(n_f + k) \times (n_f + k)$. For example, in the simplest case of quenched fermions, $N_f = 0$, the expression (6.6) reduces to

$$\rho_s(u_1, \dots, u_k) = \det_{l, l'} \left[\frac{1}{\pi} \frac{\sin(u_l - u_{l'})}{u_l - u_{l'}} \right] \quad (6.7)$$

which is the expected representation in this case of the k -point function in terms of the spectral sine-kernel of the unitary ensemble [14].

The expression (6.6) is again an explicit representation of the formula [16] for the spectral k -point function in terms of a ratio of two finite volume partition functions, one

of which now involves $2k$ additional species of fictitious quarks of imaginary mass, as

$$\begin{aligned} \rho_s(u_1, \dots, u_k; \{\omega_i\}) &= \left(\frac{1}{2\pi}\right)^k \prod_{l=1}^k \prod_{i=1}^{n_f} (u_l^2 + \omega_i^2) \Delta[u^2]^2 \\ &\times \frac{Z_{N_f+2k}^{\text{LS}}(\{\omega_i\}, iu_1, iu_1, \dots, iu_k, iu_k)}{Z_{N_f}^{\text{LS}}(\{\omega_i\})} . \end{aligned} \quad (6.8)$$

However, the representation (6.6) is a new and much more explicit form of the microscopic k -point spectral function. The equivalent representation of this function in terms of a $k \times k$ determinant of the random matrix theory spectral kernel [10, 14] leads to consistency conditions on the finite volume partition functions [16]. The very explicit determinant formula (6.6) may provide a straightforward proof of this theorem, and hence of the equivalence of this field theoretical expression with that obtained from random matrix theory. We will not do so here, but we have checked this equivalence in a number of cases. As discussed in [16], these consistency conditions are related to the fact that the finite volume gauge theory partition function, which is an Itzykson-Zuber integral, is the τ -function of an integrable KP hierarchy.

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Appendix A Supersymmetric Limit

Starting from the original random matrix theory partition function (3.1) and following the exact same steps used in section 3.1 to arrive at (3.16), we may infer the supersymmetric representation

$$Z_{N_f, N_v}(\{m_i\}; \{\mu_i, \bar{\mu}_i\}) = \int_{gl(N_f+N_v|N_v)} D\Lambda \, e^{-\frac{N\Sigma_0^2}{2} \text{STr} \Lambda^2} \, \text{SDet}^N(\Lambda - i\hat{\mathcal{M}}') . \quad (\text{A.1})$$

In this appendix we will start by formally proving that the partition function (A.1) reduces to the expected one for ordinary QCD₃ in the supersymmetric limit where $\mu_i = \bar{\mu}_i$ for each $i = 1, \dots, N_v$. For this, we decompose the $(N_f + N_v|N_v)$ supermatrix Λ into the $(1|1)$ supermatrix

$$\bar{\Lambda} = \begin{pmatrix} \lambda_{N_f+1, N_f+1} & \bar{\chi}_{N_f+1, 1} \\ \chi_{1, N_f+1} & i\bar{\lambda}_{11} \end{pmatrix} , \quad (\text{A.2})$$

the $2 \cdot (N_f + 2N_v - 2)$ supervectors

$$\begin{aligned}\Psi_A &= \begin{pmatrix} \Lambda_{A,N_f+1} \\ \Lambda_{A,N_f+N_v+1} \end{pmatrix} \\ \bar{\Psi}_A &= (\Lambda_{N_f+1,A}, \Lambda_{N_f+N_v+1,A}) \quad \text{for } A \neq N_f+1, N_f+N_v+1\end{aligned} \quad (\text{A.3})$$

of dimension $(1|1)$, and the remaining $(N_f + N_v - 1|N_v - 1)$ supermatrix which we denote by Λ_r . We may then write the integrand $F(\Lambda) = e^{-\frac{N\Sigma_0^2}{2} \text{STr} \Lambda^2} \text{SDet}^N(\Lambda - i\hat{\mathcal{M}}')$ of (A.1) as

$$F(\Lambda) = \bar{F}(\bar{\Lambda}; \{\Psi_A, \bar{\Psi}_A\}; \Lambda_r), \quad (\text{A.4})$$

and in the degenerate case $\mu_i = \bar{\mu}_i$ it possesses the invariances

$$\bar{F}(U\bar{\Lambda}U^{-1}; \{U\Psi_\alpha, \bar{\Psi}_\alpha U^{-1}\}; \Lambda_r) = \bar{F}(\bar{\Lambda}; \{\Psi_\alpha, \bar{\Psi}_\alpha\}; \Lambda_r) \quad (\text{A.5})$$

for all $U \in GL(1|1)$. The partition function may then be written as

$$Z_{N_f, N_v}(\{m_i\}; \{\mu_i, \mu_i\}) = \int_{gl(N_f+N_v-1|N_v-1)} D\Lambda_r \prod_A \int_{\mathbb{C}^{1|1}} D\Psi_A D\bar{\Psi}_A \mathcal{Z}(\{\Psi_B, \bar{\Psi}_B\}; \Lambda_r) \quad (\text{A.6})$$

where

$$\mathcal{Z}(\{\Psi_A, \bar{\Psi}_A\}; \Lambda_r) = \int_{gl(1|1)} D\bar{\Lambda} \bar{F}(\bar{\Lambda}; \{\Psi_A, \bar{\Psi}_A\}; \Lambda_r). \quad (\text{A.7})$$

Using the invariance of the Haar measure one finds that (A.7) is an invariant function of the supervectors (A.3),

$$\mathcal{Z}(\{U\Psi_A, \bar{\Psi}_A U^{-1}\}; \Lambda_r) = \mathcal{Z}(\{\Psi_A, \bar{\Psi}_A\}; \Lambda_r) \quad , \quad U \in GL(1|1). \quad (\text{A.8})$$

It then follows from the Parisi-Sourlas reduction [27] that

$$Z_{N_f, N_v}(\{m_i\}; \{\mu_i, \mu_i\}) = \int_{gl(N_f+N_v-1|N_v-1)} D\Lambda_r \mathcal{Z}(\{0, 0\}; \Lambda_r). \quad (\text{A.9})$$

Next we may invoke the Efetov-Wegner theorem [27] which states that

$$\int_{gl(1|1)} D\bar{\Lambda} G(\bar{\Lambda}) = G(0) \quad (\text{A.10})$$

for any supersymmetric invariant function G , i.e. $G(\bar{\Lambda}) = G(U\bar{\Lambda}U^{-1})$ for all $U \in GL(1|1)$. Applying this result to the integral $\mathcal{Z}(\{0, 0\}; \Lambda_r)$, we arrive at

$$Z_{N_f, N_v}(\{m_i\}; \{\mu_i, \mu_i\}) = \int_{gl(N_f+N_v-1|N_v-1)} D\Lambda_r \bar{F}(0; \{0, 0\}; \Lambda_r). \quad (\text{A.11})$$

Now we repeat this procedure by decomposing Λ_r analogously, starting with a (1|1) supermatrix $\bar{\Lambda}_r$ with diagonal bosonic elements λ_{N_f+2, N_f+2} and $i\bar{\lambda}_{22}$, as in (A.2). We iterate this reduction until all the Grassmann components and the $N_v \times N_v$ boson-boson block of the original supermatrix Λ have been eliminated. The final result is the reduced partition function

$$Z_{N_f, N_v}(\{m_i\}; \{\mu_i, \mu_i\}) = Z_{N_f, 0}(\mathcal{M}) = \int_{u(N_f)} DX \, e^{-\frac{N \Sigma_0^2}{2} \text{tr} X^2} \det^N(X - i\mathcal{M}) . \quad (\text{A.12})$$

It is independent of the valence quark masses and is the standard one for ordinary QCD₃ expressed as an integral over the physical flavour space [15]. In the microscopic limit, it becomes the finite volume partition function (1.6), $Z_{N_f, 0}^{(\infty)}(\mathcal{M}) = Z_{N_f}^{\text{LS}}(\mathcal{M})$ [9, 15]. In the original field theory formulation, this reduction can be understood in perturbation theory by noting that each bosonic superpartner to the valence fermions contributes a Feynman diagram of equal magnitude but opposite sign to the partition function. This cancellation of graphs with valence fermion loops is precisely what is required for the associated spectral density to be equal to the QCD₃ spectral density. Note that this normalization is non-trivial, since the invariances of the function $G(\bar{\Lambda})$ would naively imply that the integral (A.10) vanishes because of the Grassmann integrations. Indeed, the right-hand side of (A.10) is an example of an Efetov-Wegner boundary term which is characteristic of superintegrals [25]–[29].

A similar argument can be used to show that the supersymmetric form (3.16) of the random matrix theory partition function (3.3) reduces to (A.1) in the limit $\bar{\mathcal{M}} \rightarrow \infty$. For this, we write the superdeterminant in (3.16) as

$$\begin{aligned} \text{SDet}(\Lambda - i\hat{\mathcal{M}}) &= \prod_{j=1}^{N_f} \frac{i}{\bar{m}_j} \text{SDet} \left[i \text{diag} \left(\mathbb{1}_{N_f+N_v} \left| \bar{\mathcal{M}}^{-1}, \mathbb{1}_{N_v} \right. \right) \Lambda \right. \\ &\quad \left. + \text{diag} \left(\mathcal{M}, \mu_1, \dots, \mu_{N_v} \left| \mathbb{1}_{N_f}, \bar{\mu}_1, \dots, \bar{\mu}_{N_v} \right. \right) \right] . \end{aligned} \quad (\text{A.13})$$

We now use (3.17) to expand (A.13) for $\bar{m}_\alpha \rightarrow \infty$ in the $N_f \times N_f$ boson-boson block matrix corresponding to the physical quark superpartners. In this limit, the argument of the superdeterminant in (A.13) is a $GL(N_f, \mathbb{C})$ invariant function of this block, and also of the $2N_v$ supervectors in $\mathbb{C}^{N_f|N_f}$ which comprise the $N_f \times N_v$ and $N_v \times N_f$ blocks of Λ and their supersymmetric counterparts. By applying the above reduction theorems, we thereby find that (3.16) reduces to (A.1) in the large mass limit. The question of whether the supersymmetric and large mass limits commute with the thermodynamic limit $N \rightarrow \infty$ is somewhat more subtle. The appropriate large mass expansion for $N \rightarrow \infty$ is described in section 6, along with the pertinent Efetov-Wegner terms for the supersymmetric limit. For a more detailed proof that the limits $N \rightarrow \infty$ and $\bar{m}_\alpha \rightarrow \infty$ are commutable at least for the purpose of computing spectral correlation functions, see [37], where a similar supersymmetrization using fictitious supersymmetric sea quark partners was applied to the partition function of the chiral Gaussian unitary ensemble relevant for QCD₄.

Appendix B Supersymmetric Itzykson-Zuber Formula

In this appendix we will give a simple derivation of the supersymmetric extension of the Itzykson-Zuber formula for the unitary supergroup $U(N|M)$ [22]. The integral is

$$\mathcal{I}[X, Y; \kappa] = \int_{U(N|M)} DU \, e^{\kappa \text{STr}(XUYU^\dagger)} \quad (\text{B.1})$$

where

$$\begin{aligned} DU &= \prod_{i,j=1}^N dU_{ij} dU_{ij}^* \delta\left(\sum_A U_{iA} U_{jA}^* - \delta_{ij}\right) \prod_{\alpha,\beta=1}^M dU_{\alpha\beta} dU_{\alpha\beta}^* \delta\left(\sum_A U_{\alpha A} U_{\beta A}^* - \delta_{\alpha\beta}\right) \\ &\otimes \prod_{k=1}^N \prod_{\sigma=1}^M \frac{\partial}{\partial U_{k\sigma}} \frac{\partial}{\partial U_{k\sigma}^*} \sum_{B=1}^{N+M} U_{kB} U_{\sigma B}^* \prod_{\rho=1}^N \prod_{l=1}^M \frac{\partial}{\partial U_{\rho l}} \frac{\partial}{\partial U_{\rho l}^*} \sum_{C=1}^{N+M} U_{\rho C} U_{lC}^* \end{aligned} \quad (\text{B.2})$$

is the invariant Haar-Berezin measure on $U(N|M)$, $\kappa \in \mathbb{C}$ is a constant parameter, and $X, Y \in u(N|M)$. The super-unitary invariance of the Haar measure implies that X and Y may be taken to be diagonal without any loss of generality. Let $x_A = (x_i, \bar{x}_\alpha)$ and $y_A = (y_i, \bar{y}_\alpha)$ be their respective supereigenvalues, where here and in the following capital Latin letters $A = 1, \dots, N+M$ label the complete set of supersymmetric indices, while lower case Latin letters $i = 1, \dots, N$ label the fermionic indices and Greek letters $\alpha = 1, \dots, M$ the bosonic indices (This is the same notation used in the text). We denote by $\varepsilon(A)$ the Grassmann grading of the index A defined by $\varepsilon(i) = 1 \bmod 2$ and $\varepsilon(\alpha) = 0 \bmod 2$.

The crucial observation that we shall make here is that the integral (B.1) is effectively defined over the homogeneous superspace $U(N|M)/U(1)^{N|M}$. In the purely bosonic case $M = 0$, such an integral would be over a coadjoint orbit of the ordinary unitary group and it would define a dynamical system which satisfies the hypotheses of the Duistermaat-Heckman theorem (see [38] for a detailed exposition of the subject). This means that the semi-classical approximation to the integral, obtained by summing over all extrema (minima, maxima and saddle-points) of the argument of the exponential, is exact. In the present case, we need the analog of the Duistermaat-Heckman theorem for supermanifolds. The conditions under which the stationary phase approximation is exact for such superintegrals are discussed in [39]. In the following we will assume that these criteria are met by the integral (B.1) and simply evaluate it in the saddle-point approximation. This is justified by the fact that the ordinary integration manifolds supporting the unitary supergroup are symmetric spaces, in the usual sense. The basic point is that the superintegral (B.1) contains the same, large amount of symmetries that its bosonic counterpart has, which is the feature that is always responsible for the exactness of the semi-classical approximation in such instances [38].

For this, we first need to find the extrema of the function

$$\mathcal{H}[U] = \text{STr} \left(X U Y U^\dagger \right) = \sum_{A,B=1}^{N+M} (-1)^{\varepsilon(A)+1} x_A y_B \left| U_{AB} \right|^2. \quad (\text{B.3})$$

By using the identity

$$\frac{\partial}{\partial U_{AB}} U_{CD}^\dagger = (-1)^{\varepsilon(B)+\varepsilon(C)+1} U_{CA}^\dagger U_{BD}^\dagger \quad (\text{B.4})$$

for $U \in U(N|M)$, we find that the saddle-point equation reads

$$\left[X, U Y U^\dagger \right] = 0 \quad (\text{B.5})$$

which for X and Y diagonal becomes

$$x_A \sum_{C=1}^{N+M} U_{AC} y_C U_{CB}^\dagger = (-1)^{\varepsilon(A)+\varepsilon(B)} x_B \sum_{C=1}^{N+M} U_{AC} y_C U_{CB}^\dagger. \quad (\text{B.6})$$

Up to irrelevant elements of the Cartan subgroup $U(1)^{N|M}$ of $U(N|M)$, the only solutions U of (B.6) are those matrices which permute the eigenvalues of the matrix Y , i.e. $\sum_C U_{AC} y_C U_{CB}^\dagger = y_{P(A)} \delta_{AB}$ with $P \in S_{N+M}$. It is easy to see that there are no Grassmann-odd permutation matrices $P \in U(N|M)$ that can map bosonic and fermionic indices into one another, i.e. for which $i = P(\alpha)$. Therefore, only Grassmann-even P can occur and they take the generic form

$$P_{AB} = \begin{pmatrix} \Pi_{ij} & 0 \\ 0 & \bar{\Pi}_{\alpha\beta} \end{pmatrix} \quad \text{with } \Pi \in S_N, \bar{\Pi} \in S_M. \quad (\text{B.7})$$

The saddle-point approximation thereby dictates to sum over all elements of the discrete Weyl subgroup of the ordinary Lie group supporting $U(N|M)$.

We now set $U = (\Pi \oplus \bar{\Pi}) e^{iL}$ in (B.3), with $L \in u(N|M)$ an infinitesimal Hermitian supermatrix, and expand the function $\mathcal{H}[U]$ to quadratic order in L . The semi-classical approximation to the unitary supermatrix integral (B.1), obtained by summing over all extrema, thereby reads

$$\begin{aligned} \mathcal{I}[X, Y; \kappa] &= \frac{1}{N!M!} \sum_{\Pi \in S_N} \sum_{\bar{\Pi} \in S_M} \exp \kappa \left(\sum_{i=1}^N x_i y_{\Pi(i)} - \sum_{\alpha=1}^M \bar{x}_\alpha \bar{y}_{\bar{\Pi}(\alpha)} \right) \\ &\times \int_{u(N)} \prod_{i,j=1}^N dL_{ij} \int_{u(M)} \prod_{\alpha,\beta=1}^M dL_{\alpha\beta} \otimes \prod_{k=1}^N \prod_{\sigma=1}^M \frac{\partial}{\partial L_{i\alpha}} \frac{\partial}{\partial L_{\sigma i}} \\ &\times \exp \kappa \left[\frac{1}{2} \sum_{i,j=1}^N \left| L_{ij} \right|^2 (x_i - x_j) (y_{\Pi(i)} - y_{\Pi(j)}) \right. \\ &- \frac{1}{2} \sum_{\alpha,\beta=1}^M \left| L_{\alpha\beta} \right|^2 (\bar{x}_\alpha - \bar{x}_\beta) (\bar{y}_{\bar{\Pi}(\alpha)} - \bar{y}_{\bar{\Pi}(\beta)}) \\ &\left. + \sum_{i=1}^N \sum_{\alpha=1}^M \left| L_{i\alpha} \right|^2 (x_i - \bar{x}_\alpha) (y_{\Pi(i)} - \bar{y}_{\bar{\Pi}(\alpha)}) \right]. \quad (\text{B.8}) \end{aligned}$$

By evaluating the Gaussian integrals over the complex bosonic variables L_{ij} , $i \neq j$ and $L_{\alpha\beta}$, $\alpha \neq \beta$, the real bosonic variables L_{ii} and $L_{\alpha\alpha}$, and the complex Grassmann variables $L_{i\alpha}$, we arrive at

$$\begin{aligned} \mathcal{I}[X, Y; \kappa] &= \frac{1}{N!M!} \sum_{\Pi \in S_N} \sum_{\bar{\Pi} \in S_M} \prod_{i=1}^N e^{\kappa x_i y_{\Pi(i)}} \prod_{\alpha=1}^M e^{-\kappa \bar{x}_\alpha \bar{y}_{\bar{\Pi}(\alpha)}} \\ &\times \left(\frac{2\pi}{\kappa} \right)^{\frac{N(N-1)}{2}} \frac{\text{sgn } \Pi}{\Delta[x]\Delta[y]} \left(-\frac{2\pi}{\kappa} \right)^{\frac{M(M-1)}{2}} \frac{\text{sgn } \bar{\Pi}}{\Delta[\bar{x}]\Delta[\bar{y}]} \kappa^{NM} \Delta[x, \bar{x}] \Delta[y, \bar{y}] \end{aligned} \quad (\text{B.9})$$

where

$$\Delta[\lambda] = \det_{i,j} [\lambda_i^{j-1}] = \prod_{i>j} (\lambda_i - \lambda_j) \quad (\text{B.10})$$

is the Vandermonde determinant, and

$$\Delta[\lambda, \bar{\lambda}] = \prod_{i=1}^N \prod_{\alpha=1}^M (\lambda_i - \bar{\lambda}_\alpha) . \quad (\text{B.11})$$

In arriving at (B.9) we have used the properties

$$\begin{aligned} \prod_{i>j} (\lambda_{\Pi(i)} - \lambda_{\Pi(j)}) &= \text{sgn}(\Pi) \Delta[\lambda] , \\ \prod_{i=1}^N \prod_{\alpha=1}^M (\lambda_{\Pi(i)} - \bar{\lambda}_{\bar{\Pi}(\alpha)}) &= \Delta[\lambda, \bar{\lambda}] . \end{aligned} \quad (\text{B.12})$$

Summing over the permutations in (B.9) then leads to

$$\begin{aligned} \mathcal{I}[X, Y; \kappa] &= \frac{(2\pi)^{\frac{N(N-1)}{2} + \frac{M(M-1)}{2}}}{N!M!} \kappa^{-\frac{N(N-1)}{2}} (-\kappa)^{\frac{M(M-1)}{2}} \kappa^{NM} \\ &\times \Delta[x, \bar{x}] \Delta[y, \bar{y}] \frac{\det_{i,j} [e^{\kappa x_i y_j}]}{\Delta[x]\Delta[y]} \frac{\det_{\alpha,\beta} [e^{-\kappa \bar{x}_\alpha \bar{y}_\beta}]}{\Delta[\bar{x}]\Delta[\bar{y}]} \end{aligned} \quad (\text{B.13})$$

which is the standard supersymmetric generalization of the Itzykson-Zuber formula [22].⁵

The above derivation can be extended in a straightforward fashion to the generalization of (B.1) over any connected, compact, semi-simple Lie supergroup G , with X and Y elements of the Cartan subalgebra of the Lie superalgebra of G . By extending the formalism described at length in [38], one may in this way derive the appropriate supersymmetric extension of the Harish-Chandra formula [40] in the form conjectured in [33]. The present method gives a much simpler and compact way of deriving these supersymmetric integration formulas, in contrast to the superspace heat kernel and supergroup theoretic methods employed in [22].

⁵The expression (B.13) agrees with those obtained in [22] up to the overall numerical prefactor which is related to the volume of the ordinary unitary group supporting $U(N|M)$ in the bosonic Haar measure and also the volume of the corresponding Weyl subgroup.

References

- [1] R. Jackiw and S. Templeton, Phys. Rev. **D23** (1981) 2291.
- [2] P.B. Wiegmann, Phys. Rev. Lett. **60** (1988) 821.
- [3] J. Cornwall, Phys. Rev. **D22** (1980) 1452;
R.D. Pisarski, Phys. Rev. **D29** (1984) 2423.
- [4] A.N. Redlich, Phys. Rev. Lett. **52** (1984) 18; Phys. Rev. **D29** (1984) 2366.
- [5] P.H. Damgaard, U.M. Heller, A. Krasnitz and T. Madsen, Phys. Lett. **B440** (1998) 129, [hep-lat/9803012].
- [6] C. Vafa and E. Witten, Nucl. Phys. **B234** (1984) 173.
- [7] T. Banks and A. Casher, Nucl. Phys. **B169** (1980) 103.
- [8] H. Leutwyler and A. Smilga, Phys. Rev. **D46** (1992) 5607.
- [9] J.J.M. Verbaarschot and I. Zahed, Phys. Rev. Lett. **73** (1994) 2288, [hep-th/9405005].
- [10] P.H. Damgaard and S.M. Nishigaki, Phys. Rev. **D57** (1998) 5299, [hep-th/9711096].
- [11] J.J.M. Verbaarschot and T. Wettig, [hep-ph/0003017], to appear in Ann. Rev. Nucl. Part. Sci.
- [12] E. Brézin and A. Zee, Nucl. Phys. **B402** (1993) 613.
- [13] M.A. Halasz and J.J.M. Verbaarschot, Phys. Rev. **D52** (1995) 2563, [hep-th/9502096];
G. Akemann, P.H. Damgaard, U. Magnea and S.M. Nishigaki, Nucl. Phys. **B487** (1997) 721 [hep-th/9609174];
J. Christiansen, Nucl. Phys. **B547** (1999) 329, [hep-th/9809194];
T. Nagao and S.M. Nishigaki, [hep-th/0005077], to appear in Phys. Rev. **D62**.
- [14] G. Akemann and P.H. Damgaard, Nucl. Phys. **B528** (1998) 411, [hep-th/9801133].
- [15] U. Magnea, Phys. Rev. **D61** (2000) 056005, [hep-th/9907096]; Phys. Rev. **D62** (2000) 016005, [hep-th/9912207].
- [16] G. Akemann and P.H. Damgaard, Nucl. Phys. **B576** (2000) 597, [hep-th/9910190].
- [17] J.C. Osborn, D. Toublan and J.J.M. Verbaarschot, Nucl. Phys. **B540** (1999) 317, [hep-th/9806110];
P.H. Damgaard, J.C. Osborn, D. Toublan and J.J.M. Verbaarschot, Nucl. Phys. **B547** (1999) 305, [hep-th/9811212];
D. Toublan and J.J.M. Verbaarschot, Nucl. Phys. **B560** (1999) 259, [hep-th/9904199].
- [18] C. Bernard and M.F.L. Golterman, Phys. Rev. **D46** (1992) 853, [hep-lat/9204007];
Phys. Rev. **D49** (1994) 486, [hep-lat/9306005];
M.F.L. Golterman, Acta Phys. Polon. **B25** (1994) 1731, [hep-lat/9411005].
- [19] J.J.M. Verbaarschot, Phys. Lett. **B368** (1996) 137, [hep-ph/9509369].
- [20] J.J.M. Verbaarschot, H.A. Weidenmüller and M.R. Zirnbauer, Phys. Rep. **129** (1985) 367.

- [21] F.A. Berezin, *Introduction to Superanalysis* (Reidel, Dordrecht, 1987).
- [22] T. Guhr, J. Math. Phys. **32** (1991) 336;
J. Alfaro, R. Medina and L.F. Urrutia, J. Math. Phys. **36** (1995) 3085, [hep-th/9412012];
T. Guhr, Commun. Math. Phys. **176** (1996) 555;
T. Guhr and T. Wettig, J. Math. Phys. **37** (1996) 6395, [hep-th/9605110].
- [23] S. Coleman and E. Witten, Phys. Rev. Lett. **45** (1980) 100.
- [24] G. 't Hooft, Nucl. Phys. **B72** (1974) 461.
- [25] M.R. Zirnbauer, J. Math. Phys. **37** (1996) 4986, [math-ph/9808012].
- [26] M.R. Zirnbauer, in: *12th International Congress of Mathematical Physics*, eds. D. DeWit, A.J. Bracken, M.D. Gould and P.A. Pearse (International Press, Boston, 1999), p. 290, [chao-dyn/9810016].
- [27] G. Parisi and N. Sourlas, Phys. Rev. Lett. **43** (1979) 744;
K.B. Efetov, Adv. Phys. **32** (1983) 53;
F.J. Wegner, Z. Phys. **B49** (1983) 297;
F. Constantinescu and H.F. de Groote, J. Math. Phys. **30** (1989) 981.
- [28] M.J. Rothstein, Trans. Amer. Math. Soc. **299** (1987) 387;
T. Guhr, Nucl. Phys. **A560** (1993) 223;
J. Alfaro and L.F. Urrutia, [hep-th/9810130].
- [29] M.R. Zirnbauer, Nucl. Phys. **B265** [FS15] (1986) 375;
R. Bundschuh, Diploma Thesis, Universität zu Köln (1993), unpublished;
M.R. Zirnbauer and F.D.M. Haldane, Phys. Rev. **B52** (1995) 8729, [cond-mat/9504108];
P.-B. Gossiaux, Z. Pluhař and H.A. Weidenmüller, Ann. Phys. **268** (1998) 273, [cond-mat/9803362].
- [30] I.S. Gradshteyn and I.M. Ryzhik, *Table of Integrals, Series, and Products* (Academic Press, San Diego, 1980).
- [31] H. Jeffreys and B.S. Jeffreys, *Methods of Mathematical Physics* (Cambridge University Press, 1946).
- [32] M.L. Mehta, *Random Matrices* (Academic Press, San Diego, 1991).
- [33] M.R. Zirnbauer, J. Phys. **A29** (1996) 7113, [chao-dyn/9609007].
- [34] K.B. Efetov, *Supersymmetry in Disorder and Chaos* (Cambridge University Press, 1997).
- [35] C. Itzykson and J.-B. Zuber, J. Math. Phys. **21** (1980) 411.
- [36] T. Nagao and K. Slevin, J. Math. Phys. **34** (1993) 2075.
- [37] B. Seif, T. Wettig and T. Guhr, Nucl. Phys. **B548** (1999) 475, [hep-th/9811044].
- [38] R.J. Szabo, *Equivariant Cohomology and Localization of Path Integrals*, Lecture Notes in Physics **m63** (Springer-Verlag, Berlin-Heidelberg, 2000).
- [39] A. Schwarz and O.V. Zaboronsky, Commun. Math. Phys. **183** (1997) 463, [hep-th/9511112];
O.V. Zaboronsky, [hep-th/9611157].
- [40] Harish-Chandra, Amer. J. Math. **79** (1957) 87.